INCOMPRESSIBILITY AND GLOBAL INJECTIVITY IN SECOND-GRADIENT NON-LINEAR ELASTICITY

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We prove the energy minimizers for a broad class of physically reasonable second-gradient non-linear elastic materials satisfy weak equilibrium equations with incompressibility and/or self-contact constraints. In the case of incompressibility, this is equivalent to proving the existence of a distributional pressure, and for self-contact, we prove the existence of a measure-valued surface traction supported on the coincidence set. We find that coercivity of the second-gradient energy yields enough regularity on the minimizer to rigorously take variations within the class of incompressible and/or globally injective deformations. A major difficulty lies in constructing sufficiently regular incompressible/globally injective variations with prescribed boundary conditions. We discuss some of the necessary regularity theory with an approach that has applications to broader mixed-order non-linear elliptic systems. The admissible deformations are globally injective if Dirichlet boundary conditions are imposed on the entire boundary. However with mixed boundary conditions, minimizers may lose injectivity without an additional constraint. The self-contact constraint gives an example of an infinite-dimensional variational inequality. For the self-contact problem, a significant portion of the work goes into characterizing the tangent cone at an admissible deformation.

BIOGRAPHICAL SKETCH

Aaron Zeff Palmer was born May 28, 1988 to Zelda Barbara Zabinsky and John Clinton Palmer. Their second child after Rebecca Anne Zabinksy. Aaron went to the University of California in Santa Cruz as an undergraduate to study in math and physics. He began graduate school as Cornell University in 2011 where he was advised by Timothy J. Healey in the math department and partially funded by a NSF graduate research fellowship.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Although the basic foundation of non-linear elasticity was laid by Cauchy in the first half of the nineteenth century, fundamental questions on the existence and regularity of equilibrium solutions remain inadequately answered. The weak solution theory of linear systems of PDE, and the corresponding regularity theory, give a fairly complete resolution to the mathematical question of well-posedness for problems of small deformations (linear elasticity) [2], [21], [24]. Progress for finite deformations (non-linear elasticity) has been significantly slower. Morrey discovered how the condition of quasi-convexity implies existence of minimizers in the calculus-of-variations [31]. Then Ball recognized the sufficient geometric condition of polyconvexity [5], along with compatible conditions on the stored energy function that imply finite-energy deformations are locally injective off of a set of measure zero. With these conditions, Ball proved the existence of energy minimizers in finite elasticity under physically realistic assumptions. The key problem remains that the energy minimizers may fail to satisfy weak (variational) equations due to the singular growth of the energy necessary to enforce local injectivity.

The incompressibility constraint also implies local injectivity of deformations, and it has a long history in non-linear elasticity. Experiments have found that bulk rubber is nearly incompressible, and incompressibility has mathematical significance due to the simplification of the stress-strain relationship [32] and the plentiful existence of special equilibrium solutions [33]. As in the compressible case, Ball showed there exists an incompressible energy minimizer [5]. However, it is not known how to construct variations in the class of incompressible deformations and prove that the minimizer satisfies a weak equilibrium equation. Developments of the mathematical theory for incompressible materials, including when energy minimizers with higher regularity satisfy equilibrium equations, have been made by Le Dret [28], and LeTallec and Oden [29].

We consider a class of second-gradient non-linear elasticity models. Within this framework, we develop a consistent and physically realistic set of assumptions for which we prove the existence of incompressible weak equilibrium solutions. We emphasize the case of mixed boundary conditions, which presents interesting and surprising difficulties for compatibility with the incompressibility constraint. Different techniques are required to handle the portion of the boundary with displacement boundary conditions, which unsurprisingly requires more regularity assumptions, and the portion of the boundary with natural boundary conditions, which we only require to satisfy the cone condition. For second-gradient compressible materials, Healey and Krömer [23] assume singular growth of the stored energy as the determinant of the deformation gradient approaches zero. They prove that the Jacobian determinant of finite-energy deformations is uniformly bounded away from zero and that the energy minimizers solve the weak Euler-Lagrange equations.

With mixed boundary conditions, the deformations may self intersect unrealistically, so we impose a global injectivity constraint. This constraint is closely related to obstacle problems, which are often studied with convex admissible sets. The application to non-linear elasticity is clearly presented by Ciarlet in [11]. Ciarlet shows the existence of equilibrium solutions with the formal assumption of full regularity (at least C^2) and thus has classical solutions to the equilibrium equations and a continuously differentiable surface traction. Furthermore, Ciarlet proves the existence of almost everywhere injective energy minimizers by means of an integral constraint. However, this formulation is apparently not conducive for proving that minimizers satisfy equilibrium equations. We develop an approach for the existence of injective minimizers that solve weak equilibrium equations, under the assumptions of second-gradient non-linear elasticity for both compressible and incompressible materials. In particular, we prove the existence of a measure-valued surface traction enforcing the self-contact constraint, which includes the possibility that contact is made at a single point with a finite amount of force imparted on the material. Our approach differs significantly from the literature on variational inequalities in that we focus on the non-linear and non-convex nature of the constraint.

The incompressibility constraint and self-contact constraint differ in a manner common between equality and inequality constraints. There are a handful of benefits for working with inequality constraints. We do not require there to be a continuously differentiable mapping that imposes the self-contact constraint as we need for incompressibility. Instead, we work with properties of interior cones to the admissible set of deformations. We benefit from the additional 'room' to take variations in the interior. This translates to higher regularity of the Lagrange multiplier for the self-contact constraint. We find the surface traction is a finite vector-valued measure, and the dual pairing with interior vector fields is non-positive. In contrast, we only show the internal pressure is in a dual space that contains the finite-signed measures.

1.1 Summary

We begin Chapter 2 with a formulation of the problem and list of assumptions. We routinely show the existence of incompressible minimizers by the direct method for the calculus-of-variations. We then spend the bulk of the chapter showing that the hypothesis of the Lagrange multiplier theorem is satisfied for admissible deformations. This implies the existence of a pressure and that minimizers satisfy a weak equilibrium equation. A key component of the analysis is the embedding of the twice weakly differentiable Sobolev functions into the class of continuously differentiable functions (with Sobolev exponent $p > 3$). In the compressible case, this is used to show the Jacobian determinant is uniformly greater than zero [23]. For incompressible materials, the Sobolev embedding into continuously differentiable functions plays an essential role to show the determinant operator maps into the Sobolev space of once weakly differentiable functions with the same exponent. We construct implicitly defined volume preserving variations within the Sobolev space by applying the surjective implicit function theorem on Banach spaces. We work to show the technical requirements of continuous Fréchet differentiability of the determinant operator and surjectivity of the linearization of the determinant with the same function spaces. The largest technical challenge is to prove surjectivity of the linearized operator with compatible boundary conditions, and we cover the background regularity theory in Chapter 4.

In Chapter 3, we study the additional constraint of global injectivity. We prove that minimizers of globally injective deformations are injective on the interior of the reference configuration, utilizing the higher regularity from the second-gradient assumptions. To prove that minimizers satisfy equilibrium equations, we take an approach analogous to infinite-dimensional Karush-Kuhn-Tucker conditions for inequality constraints. We characterize a sub-cone of the interior cone to the admissible deformations by criterion on the coincidence set, and we show it has non-empty interior. With the additional assumption that the boundary is everywhere continuously differentiable, we find the sub-cone completely describes the admissible deformations locally. When we only assume a strongly Lipschitz boundary, we no longer describe all the locally admissible deformations, but we describe enough to carry out a similar proof. This analysis nicely relates the boundary properties of the domain in three-dimensional Euclidean space, and the properties of the boundary of the infinite-dimensional set of admissible deformations.

For much of this thesis we collect and combine theory of analysis and PDE in the context of non-linear second-gradient elasticity. Some of the background theory is well known while some is more obscure. Because the subtleties of many of the background results play an important role in their applications, we present and discuss some of the background at the end of the thesis. We devote Chapter 4 to the regularity theory for solutions to the linearized incompressibility constraint equation. In Appendix A we cover some useful background on Fréchet differentiability, and in particular, continuous differentiability of Nemytskii operators. In Appendix B we discuss questions related to the surjective implicit function theorem and Lagrange multiplier theorem, and answer them by giving proofs and counterexamples. Finally, Appendix C contains analysis of the boundary of Lipschitz domains.

CHAPTER 2

INCOMPRESSIBLE ENERGY MINIMIZERS AND WEAK SOLUTIONS IN SECOND-GRADIENT NON-LINEAR ELASTICITY.

2.0.1 Notation

Let $\mathbb{E}^3$ be three-dimensional Euclidean (translate) space, and $\Omega \subset \mathbb{E}^3$ an open, bounded and connected subset with (strongly) Lipschitz boundary (Definition C.2). Whenever possible we avoid reference to a choice of coordinates for $\mathbb{E}^3$, but when needed, we fix an origin and let $\{\mathbf{e}_i\}_{i=1}^3$ be an orthonormal basis and $\{x^i\}_{i=1}^3$ the corresponding coordinate functions so that $\mathbf{x} = x^i\mathbf{e}_i$ (adjacent indices are assumed to be summed unless otherwise stated).

The primary setting for this work is the Sobolev space $W^{2,p}(\Omega, \mathbb{E}^3)$. This is the space of deformations $\mathbf{f} : \Omega \to \mathbb{E}^3$ such that each coordinate is twice weakly differentiable and all the first and second partial derivatives are in $L^p(\Omega)$. Although general Sobolev spaces are defined as equivalences classes up to sets of measure zero, we always assume that $p > 3$ so that there is an embedding from $W_0^{2,p}(\Omega, \mathbb{E}^3) \to C^1(\overline{\Omega}, \mathbb{E}^3)$, and we choose the unique continuously differentiable representative of the equivalence class. The subspace $W_0^{2,p}(\Omega, \mathbb{E}^3)$ consists of functions that vanish, along with their derivatives, on the boundary of Ω. Both $W^{2,p}(\Omega, \mathbb{E}^3)$ and $W_0^{2,p}(\Omega, \mathbb{E}^3)$ are Banach spaces with the norm

$$||\mathbf{f}||_{W^{2,p}(\Omega,\mathbb{E}^3)} = \left(\int_\Omega \left[|\mathbf{f}|^p + |\nabla\mathbf{f}|^p + |\nabla^2\mathbf{f}|^p\right] dV\right)^{\frac{1}{p}}. \tag{2.1}$$

The space $W_0^{2,p}(\Omega, \mathbb{E}^3)$ is the closure of the compactly supported smooth functions, $C_c^\infty(\Omega, \mathbb{E}^3)$, with respect to the norm of (2.1). In (2.1), dV is the ordinary volume form on Euclidean space and ∇ is the spatial gradient. We use the symbol D for the derivative with respect to tensors or derivatives on functions spaces, and a subscript on D determines the variable of differentiation if it is ambiguous. In general, we use brackets to distinguish non-linear operators on function spaces, and linear operators act on the left without brackets or parentheses. We use angled brackets for the dual pairing with linear functionals. We understand the derivatives in (2.1) as linear transformations, $\nabla \mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \in L(\mathbb{E}^3)$ the space of linear endomorphisms of $\mathbb{E}^3$. The norm on $L(\mathbb{E}^3)$ is given by the Frobenius norm, $|\mathbf{F}| = \sqrt{\operatorname{tr}(\mathbf{F}^\top \mathbf{F})}$. In terms of a basis, $\mathbf{f}(\mathbf{x}) = f^i(\mathbf{x})\mathbf{e}_i$, $\mathbf{F}(\mathbf{x}) = F^i_{\ j}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}^j = \frac{\partial f^i(\mathbf{x})}{\partial x^j}\mathbf{e}_i \otimes \mathbf{e}^j$, and $|\mathbf{F}|^2 = F^{\ j}_i F^{\ j}_i$. (With the incompressibility constraint, $\mathbf{F}(\mathbf{x}) \in SL(3)$ the special linear group. However, it is important for our approach that $\mathbf{F}$ lives in a linear space.) The second derivatives are symmetric, bilinear maps, $\nabla^2 \mathbf{f}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) \in BL(\mathbb{E}^3) \equiv \{\mathbf{B} \in L(\mathbb{E}^3 \otimes \mathbb{E}^3, \mathbb{E}^3) : \mathbf{BA} = \mathbf{BA}^\top \ \forall\, \mathbf{A} \in \mathbb{E}^3 \otimes \mathbb{E}^3\}$. With respect to a basis, $\mathbf{G}(\mathbf{x}) = G^i_{\ jk}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k = \frac{\partial^2 f^i(\mathbf{x})}{\partial x^j \partial x^k}\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$ and $|\mathbf{G}|^2 = G^i_{\ jk} G^i_{\ jk}$.

The stored energy, $W : L(\mathbb{E}^3) \times BL(\mathbb{E}^3) \times \Omega \to \mathbb{R}$, is a real-valued function of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{x}$, and we make assumptions on the structure and regularity of W in Section 2.0.2. We use $D_F W \in L(\mathbb{E}^3)^*$ to denote differentiation with respect to the first component, where * denotes a linear functional. For the second component, we employ the notation $D_G W \in BL(\mathbb{E}^3)^*$. The third component represents dependence on the spatial variable. The total energy is a functional $E : W^{2,p}(\Omega, \mathbb{E}^3) \to \mathbb{R}$ given by

$$E[\mathbf{f}] = \int_{\Omega} \left[W(\nabla \mathbf{f}(\mathbf{x}), \nabla^2 \mathbf{f}(\mathbf{x}), \mathbf{x}) + g(\mathbf{f}(\mathbf{x}), \mathbf{x}) \right] dV + \int_{\partial\Omega} \gamma(\mathbf{f}(\mathbf{x}), \mathbf{x}) dS. \tag{2.2}$$

With this general form, we allow for conservative forces corresponding to $-D_f g$ in the interior and $-D_f \gamma$ on the boundary. As an example, suppose the material carries a charge with fixed inhomogeneous density $\lambda(\mathbf{x})$ and the ambient space has an electric potential $\Phi(\mathbf{y})$, in this case $g(\mathbf{f}(\mathbf{x}), \mathbf{x}) = \lambda(\mathbf{x})\Phi(\mathbf{f}(\mathbf{x}))$.

The incompressibility constraint is expressed using the operator $H : W^{2,p}(\Omega, \mathbb{E}^3) \to W^{1,p}(\Omega)$,

$$H[\mathbf{f}](\mathbf{x}) \equiv \det(\nabla \mathbf{f}(\mathbf{x})) = 1. \tag{2.3}$$

In the following lemma we demonstrate the precise relationship between the PDE (2.3) and the geometric notion of an orientation and volume-preserving diffeomorphism.

Lemma 2.1. *Suppose $\Omega \subset \mathbb{E}^3$ is open and bounded and $\partial\Omega$ is Lipschitz. A deformation, $\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3)$, is an orientation and volume-preserving diffeomorphism onto its image, $\mathbf{f}(\overline{\Omega})$, if and only if $\mathbf{f}$ is injective on $\overline{\Omega}$ and $H[\mathbf{f}](\mathbf{x}) = 1$ for all $\mathbf{x} \in \overline{\Omega}$.*

By volume-preserving diffeomorphism, we mean $\mathbf{f} \in C^1(\overline{\Omega}, \mathbb{E}^3)$, which is a bijection of $\overline{\Omega}$ and $\mathbf{f}(\overline{\Omega})$ with $\mathbf{f}^{-1} \in C^1(\mathbf{f}(\overline{\Omega}), \mathbb{E}^3)$, such that for all open sets $O \subset \mathbb{E}^3$, $\mathrm{Vol}(O \cap \overline{\Omega}) = \mathrm{Vol}(\mathbf{f}(O \cap \overline{\Omega}))$. This lemma relies on the Sobolev embedding of $W^{2,p}(\Omega, \mathbb{E}^3)$ into $C^1(\overline{\Omega}, \mathbb{E}^3)$ and the area formula.

Proof. If $\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3)$ is a volume-preserving diffeomorphism, then for any open

$O \subset \mathbb{E}^3$, by the area formula

$$\mathrm{Vol}(O \cap \overline{\Omega}) = \mathrm{Vol}(\mathbf{f}(O \cap \overline{\Omega})) = \int_{O \cap \overline{\Omega}} \left| \det\left(\nabla \mathbf{f}(\mathbf{x})\right) \right| dV, \tag{2.4}$$

and this implies that $|\det(\nabla \mathbf{f})(\mathbf{x})| = 1$ for every $\mathbf{x} \in \overline{\Omega}$ by continuity of the determinant of $\nabla \mathbf{f}$. Since $\mathbf{f}$ preserves orientation, we conclude that $H[\mathbf{f}](\mathbf{x}) = 1$.

In the other direction, the inverse of $\mathbf{f}$ is continuously differentiable as a consequence of the inverse function theorem. That $\mathbf{f}$ preserves volume follows immediately from (2.4), and $\mathbf{f}$ preserves orientation because $\det(\nabla \mathbf{f}(\mathbf{x})) > 0$. □

2.0.2 Problem Formulation

We make a few assumptions on the domain and the stored energy in order to formulate an appropriate calculus-of-variations problem.

A1 There exists a constant $C_c > 0$, independent of $\mathbf{F}$ or $\mathbf{x}$, and $p > 3$, such that $|\mathbf{G}|^p \leq C_c W(\mathbf{F}, \mathbf{G}, \mathbf{x})$.

A2 We suppose that $\mathbf{F} \mapsto W$ and $\mathbf{G} \mapsto W$ are C^1 for a.e. $\mathbf{x} \in \Omega$, and we also assume that W along with $D_F W$ and $D_G W$ are measurable with respect to $\mathbf{x}$, i.e. each is a Carathéodory function. Furthermore, the following growth conditions are satisfied,

$$\begin{aligned} W(\mathbf{F}, \mathbf{G}, \mathbf{x}) &\leq C_{G1}(\mathbf{F})(|\mathbf{G}|^p + 1), \\ |D_G W(\mathbf{F}, \mathbf{G}, \mathbf{x})| &\leq C_{G2}(\mathbf{F})(|\mathbf{G}|^{p-1} + 1), \end{aligned}$$

where $C_{G1}(\mathbf{F})$ and $C_{G2}(\mathbf{F})$ are bounded on compact sets.

A3 We similarly assume $\mathbf{f} \mapsto g$ and $\mathbf{f} \mapsto \gamma$ are C^1 a.e. and g, $D_f g$, γ, and $D_f \gamma$ are measurable with respect to $\mathbf{x}$. We also assume that g and γ are uniformly bounded, i.e. $|g(\mathbf{f}, \mathbf{x})|, |\gamma(\mathbf{f}, \mathbf{x})| \leq C_u$.

We need a definition in order to proceed with our assumptions.

Definition 2.1. *For a real valued map on the space of linear transformations of two finite vector spaces, $J : L(V_1, V_2) \to \mathbb{R}$, we say that J is* polyconvex *if J may be expressed as convex function of minor determinants of the transformation. Given a basis $\{\mathbf{e}^i\}_{i=1}^{n_1}$ for V_1 and $\{\mathbf{e}_j\}_{j=1}^{n_2}$ for V_2, then for sets of indices $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$, we define $\mathbf{A}_I^J = \sum_{i \in I} \sum_{j \in J} A_i^j \mathbf{e}_j \otimes \mathbf{e}^i$. Then polyconvexity may be expressed as $J(\mathbf{A}) = \tilde{J}(\det(\mathbf{A}_{I_1}^{J_1}), \det(\mathbf{A}_{I_2}^{J_2}), \ldots)$ where I_l, J_l run over all combinations of indices with $1 \leq |I_l| = |J_l| \leq \min\{n_1, n_2\}$, and $\tilde{J}$ is convex.*

Remark 2.1. *Let $\Lambda^r V_1$ denote the anti-symmetric rank r tensors of V_1. If $V_1 = V_2 = \mathbb{E}^3$, then polyconvexity is equivalent to $J(\mathbf{A}) = \tilde{J}(\mathbf{A}, \mathrm{cof}(\mathbf{A}), \det(\mathbf{A}))$ where $\tilde{J}$ is convex on $L(\mathbb{E}^3) \times L(\mathbb{E}^3) \times \mathbb{R}$. In this case $\Lambda^2 \mathbb{E}^3$ is identified with $\mathbb{E}^3$ and $\Lambda^3 \mathbb{E}^3$ is one-dimensional. For the second-gradient, we take $V_1 = \mathbb{E}^3$ and $V_2 = L(\mathbb{E}^3)$ and consider $BL(\mathbb{E}^3) \subset L(V_1, V_2)$ by $(\mathbf{G}\mathbf{v}_1)\mathbf{v}_2 = \mathbf{G}(\mathbf{v}_1 \otimes \mathbf{v}_2)$. Then $\tilde{J}$ is allowed to depend on $\mathbf{G}$, $\mathbf{G}^{[2]} \in L(\Lambda^2 \mathbb{E}^3, \Lambda^2 L(\mathbb{E}^3))$, a 108-dimensional space, and $\mathbf{G}^{[3]} \in L(\Lambda^3 \mathbb{E}^3, \Lambda^3 L(\mathbb{E}^3)) \cong \Lambda^3 L(\mathbb{E}^3)$, an 84-dimensional space. With respect to a basis for $\mathbb{E}^3$ and $L(\mathbb{E}^3)$, the coefficients of $\mathbf{G}^{[2]}$ are determinants of the 2×2 sub-matrices and $\mathbf{G}^{[3]}$ are determinants of the 3×3 sub-matrices. If we let $\wedge$ denote the anti-symmetric wedge product, for example $\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})$ or $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \frac{1}{6}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} - \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a} - \mathbf{c} \otimes \mathbf{b} \otimes \mathbf{a})$, then we* *may define* $\mathbf{G}^{[r]}$ *without reference to a basis by*

$$\mathbf{G}^{[2]}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{G}\mathbf{a}) \wedge (\mathbf{G}\mathbf{b})$$
$$\mathbf{G}^{[3]}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = (\mathbf{G}\mathbf{a}) \wedge (\mathbf{G}\mathbf{b}) \wedge (\mathbf{G}\mathbf{c}).$$

A4 We assume that $\mathbf{G} \mapsto W$ is polyconvex for each $\mathbf{F}$ and almost every $\mathbf{x}$. Explicitly, $W(\mathbf{F}, \mathbf{G}, \mathbf{x}) = \tilde{W}(\mathbf{F}, \mathbf{G}, \mathbf{G}^{[2]}, \mathbf{G}^{[3]}, \mathbf{x})$ and $(\mathbf{G}, \mathbf{G}^{[2]}, \mathbf{G}^{[3]}) \mapsto \tilde{W}$ is convex.

There are three different boundary conditions for which we define a class of admissible deformations and list a couple additional assumptions. We assume that the displacement boundary conditions are imposed by an orientation and volume-preserving diffeomorphism $\mathbf{f}_0 \in W^{2,p}(\Omega, \mathbb{E}^3)$.

S The strong Dirichlet case requires no additional assumption on $\partial\Omega$ or W. The class of admissible deformations is

$$\mathcal{A}_{in}^{S} = \Big\{\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3) : \mathbf{f} - \mathbf{f}_0 \in W_0^{2,p}(\Omega, \mathbb{E}^3),\ H[\mathbf{f}] = 1\Big\}.$$

W For the weak Dirichlet problem, we assume that $\partial\Omega$ is $W^{2,p}$ (defined in Chapter 4, Definition 4.1), and the admissible deformations are

$$\mathcal{A}_{in}^{W} = \Big\{\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3) : (\mathbf{f} - \mathbf{f}_0)(\mathbf{x}) = \mathbf{0}\ \forall\ \mathbf{x} \in \partial\Omega,\ H[\mathbf{f}] = 1\Big\}.$$

M For the mixed Dirichlet and Neumann problem, we let $\Gamma \subset \partial\Omega$ be a closed non-empty set on which the weak Dirichlet boundary conditions are imposed. We assume that Γ is contained in a $W^{2,p}$ surface. This means that for each

$\mathbf{x} \in \Gamma$ there is a neighborhood O and a diffeomorphism $\boldsymbol{\phi} \in W^{2,p}(O, \mathbb{E}^3)$ such that

$$\boldsymbol{\phi}(\Gamma \cap O) \subset \{\mathbf{y} : \mathbf{y} \cdot \mathbf{e}_3 = 0\}.$$

The admissible deformations are

$$\mathcal{A}_{in}^{M} = \left\{\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3) : (\mathbf{f} - \mathbf{f}_0)(\mathbf{x}) = \mathbf{0} \ \forall\ \mathbf{x} \in \Gamma,\ H[\mathbf{f}] = 1\right\}.$$

We also need an additional assumption for a Poincaré type inequality. It is sufficient to assume that any linear displacement, i.e. $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, which vanishes on Γ, is trivial. Equivalently, there are 4 non-coplanar points in Γ.

Remark 2.2. *For elasticity, if the current configuration (the co-domain of* $\mathbf{f}$*) undergoes a rigid rotation there should be no change in the stored energy. This is expressed as the material objectivity condition:* $W(\mathbf{OF}, \mathbf{OG}, \mathbf{x}) = W(\mathbf{F}, \mathbf{G}, \mathbf{x})$ *for all* $\mathbf{O} \in SO(3)$. *Material objectivity is achieved if we assume that* $W(\mathbf{F}, \mathbf{G}, \mathbf{x}) = \Phi(\mathbf{F}^\top\mathbf{F}, \mathbf{F}^\top\mathbf{G}, \mathbf{x})$. *In fact, material objectivity implies W must have this form. Consider the polar decomposition of* $\mathbf{F} = \mathbf{RU}$, *where* $\mathbf{U} = \sqrt{\mathbf{F}^\top\mathbf{F}}$ *and* $\mathbf{R} \in SO(3)$. *Then* $W(\mathbf{F}, \mathbf{G}, \mathbf{x}) = W(\mathbf{U}, \mathbf{R}^{-1}\mathbf{G}, \mathbf{x}) = W(\mathbf{U}, \mathbf{U}\mathbf{F}^{-1}\mathbf{G}, \mathbf{x}) = W(\mathbf{U}, \mathbf{U}^{-1}\mathbf{F}^\top\mathbf{G}, \mathbf{x})$ *and defining* $\Phi(\mathbf{C}, \mathbf{N}, \mathbf{x}) = W(\sqrt{\mathbf{C}}, \sqrt{\mathbf{C}}^{-1}\mathbf{N}, \mathbf{x})$ *achieves the desired representation where* $\mathbf{C} = \mathbf{F}^\top\mathbf{F}$ *and* $\mathbf{N} = \mathbf{F}^\top\mathbf{G}$.

2.1 Existence

We prove the following existence proposition using standard techniques of the calculus-of-variations for static fields.

Proposition 2.1. *Let* $\mathbf{I} \in \{\mathbf{S}, \mathbf{W}, \mathbf{M}\}$ *and suppose* W*,* g *and* γ *satisfy the assumptions of section 2.0.2. Let* E *be defined as in (2.2). Then there exists* $\mathbf{f}^* \in \mathcal{A}_{in}^I$ *such that* $E[\mathbf{f}^*] \leq E[\mathbf{f}]$ *for all* $\mathbf{f} \in \mathcal{A}_{in}^I$*.*

The proof follows from two facts on the energy E and the set of admissible displacements $\mathcal{A}_{in}^I$, which we prove in Lemmas 2.2 and 2.3.

Proof. Let $\mathbf{f}_i \in \mathcal{A}_{in}^I$ be an energy infimizing sequence. Lemma 2.2 implies that there is a uniform bound on the $W^{2,p}(\Omega, \mathbb{E}^3)$ norm of $\mathbf{f}_i$. The Banach-Alaoglu theorem states that bounded and weakly closed sets are compact with respect to the weak topology (the Sobolev spaces are reflexive so there is only one weak topology). By compactness, there is a subsequence converging weakly, $\mathbf{f}_{i_n} \rightharpoonup \mathbf{f}^*$ as $n \to \infty$. Lemma 2.3 shows that $\mathcal{A}_{in}^I$ is closed with respect to the weak topology so that $\mathbf{f}^* \in \mathcal{A}_{in}^I$. Lemma 2.3 also proves that E is weakly lower semi-continuous, which implies $E[\mathbf{f}^*] \leq \liminf_{n\to\infty} E[\mathbf{f}_{i_n}]$. Since the sequence is infimizing, we conclude that $\mathbf{f}^*$ is a minimizer. □

Lemma 2.2. *Let* $\mathbf{I} \in \{\mathbf{S}, \mathbf{W}, \mathbf{M}\}$ *and suppose* W *satisfies the conditions in section 2.0.2. There is a constant* C*, independent of* $\mathbf{f}$ *although possibly dependent on* $\mathbf{f}_0$*, such that for all* $\mathbf{f} \in \mathcal{A}_{in}^I$*,*

$$||\mathbf{f}||^p_{W^{2,p}(\Omega,\mathbb{E}^3)} \leq C(E[\mathbf{f}] + 1). \tag{2.5}$$

Proof. We have assumed that Ω is bounded, thus has finite measure, and that $\partial\Omega$ is Lipschitz, which implies the 2D Hausdorff measure of $\partial\Omega$ is bounded. Assumption **A**3 states that g and γ are uniformly bounded, so together with the coercivity assumption, **A**1, we have

$$\begin{aligned}
\|\nabla^2\mathbf{f}\|^p_{L^p(\Omega,BL(\mathbb{E}^3))} \le &\int_\Omega C_c W(\nabla\mathbf{f}(\mathbf{x}),\nabla^2\mathbf{f}(\mathbf{x}),\mathbf{x})dV\\
\le &C_c E[\mathbf{f}] + C_c C_u(|\Omega| + |\partial\Omega|)\\
\le &C_1(E[\mathbf{f}]+1).
\end{aligned}$$

In the case that $\mathbf{I} = \mathbf{S}$ the triangle inequality and Poincaré inequality imply that

$$\begin{aligned}
\|\mathbf{f}\|^p_{W^{2,p}(\Omega,\mathbb{E}^3)} &\le \left(\|\mathbf{f}-\mathbf{f}_0\|_{W^{2,p}(\Omega,\mathbb{E}^3)} + \|\mathbf{f}_0\|_{W^{2,p}(\Omega,\mathbb{E}^3)}\right)^p\\
&\le \left(C_p\|\nabla^2(\mathbf{f}-\mathbf{f}_0)\|_{L^p(\Omega,BL(\mathbb{E}^3))} + \|\mathbf{f}_0\|_{W^{2,p}(\Omega,\mathbb{E}^3)}\right)^p\\
&\le \left(C_p\|\nabla^2\mathbf{f}\|_{L^p(\Omega,BL(\mathbb{E}^3))} + (C_p+1)\|\mathbf{f}_0\|_{W^{2,p}(\Omega,\mathbb{E}^3)}\right)^p\\
&\le \left(C_p C_1^{1/p}(E[\mathbf{f}]+1)^{1/p} + (C_p+1)\|\mathbf{f}_0\|_{W^{2,p}(\Omega,\mathbb{E}^3)}\right)^p\\
&\le 2^{p-1}\left(C_p^p C_1(E[\mathbf{f}]+1) + (C_p+1)^p\|\mathbf{f}_0\|^p_{W^{2,p}(\Omega,\mathbb{E}^3)}\right), \qquad (2.6)
\end{aligned}$$

with the last step by Jensen's inequality. The same proof works for the other boundary conditions with different versions of the Poincaré inequality.

For $\mathbf{I} = \mathbf{W}$, let $\mathbf{u} = \mathbf{f} - \mathbf{f}_0$ and the average gradient is zero, $\int_\Omega \nabla\mathbf{u}\, dV = \int_{\partial\Omega} \mathbf{u}\otimes\mathbf{n}\, dS = \mathbf{0}$. Thus the Poincaré inequality for displacements that vanish on the boundary shows that $\|\mathbf{u}\|_{L^p(\Omega,\mathbb{E}^3)} \le C'_p\|\nabla\mathbf{u}\|_{L^p(\Omega,L(\mathbb{E}^3))}$. The Poincaré inequality for functions with zero mean value implies $\|\nabla\mathbf{u}\|_{L^p(\Omega,L(\mathbb{E}^3))} \le C''_p\|\nabla^2\mathbf{u}\|_{L^p(\Omega,BL(\mathbb{E}^3))}$ and the result follows as in (2.6).

For the mixed boundary case, $\mathbf{I} = \mathbf{M}$, the assumption that any linear displacements that vanish on Γ are trivial implies the existence of a Poincaré inequality.

Suppose to the contrary that there were a sequence of displacements, $\mathbf{f}_i$, which vanish on Γ, and

$$||\mathbf{f}_i||_{W^{1,p}(\Omega,\mathbb{E}^3)} \geq i||\nabla^2 \mathbf{f}_i||_{L^p(\Omega,BL(\mathbb{E}^3))}.$$

Then a standard argument, see [13], shows that $\mathbf{u}_i = \mathbf{f}_i/||\mathbf{f}_i||_{W^{1,p}(\Omega,\mathbb{E}^3)}$ has a subsequence converging strongly in $W^{1,p}(\Omega,\mathbb{E}^3)$ to $\mathbf{u}$ with $||\mathbf{u}||_{W^{1,p}(\Omega,\mathbb{E}^3)} = 1$. Then we argue that $\mathbf{u}$ has zero weak second derivatives, hence is affine linear. By the assumption of $\mathbf{M}$, $\mathbf{u} = \mathbf{0}$, which is a contradiction. So there is a constant such that, if $\mathbf{f} \in W^{2,p}(\Omega,\mathbb{E}^3)$ vanishes on Γ, then

$$||\mathbf{f}||_{W^{1,p}(\Omega,\mathbb{E}^3)} \leq C_\Gamma ||\nabla^2 \mathbf{f}||_{L^p(\Omega,BL(\mathbb{E}^3))}.$$

□

Lemma 2.3. *Let* $\mathbf{I} \in \{\mathbf{S},\mathbf{W},\mathbf{M}\}$ *and suppose* W *satisfies the conditions in section 2.0.2. Then* $\mathcal{A}^I_{in}$ *is closed in the weak topology of* $W^{2,p}(\Omega,\mathbb{E}^3)$ *and the non-linear functional* E *is weakly lower semi-continuous.*

Proof. Polyconvexity, $\mathbf{A}4$, implies that $W(\mathbf{F},\mathbf{G},\mathbf{x}) = \tilde{W}(\mathbf{F},\mathbf{G},\mathbf{G}^{[2]},\mathbf{G}^{[3]},\mathbf{x})$, and $\tilde{W}$ is convex for fixed $\mathbf{F}$ and $\mathbf{x}$. Suppose that $\mathbf{f}_i \rightharpoonup \mathbf{f}$ in $W^{2,p}$ then for each coordinate function $\frac{\partial}{\partial x^k}\mathbf{f}_i \rightharpoonup \frac{\partial}{\partial x^k}\mathbf{f}$ in $W^{1,p}(\Omega,\mathbb{E}^3)$. A well known result on weak convergence, [5], implies all the minor determinants (of the transformation viewed in $L(\mathbb{E}^3, L(\mathbb{E}^3))$ converge, $\mathbf{G}^{[r]}_i \rightharpoonup \mathbf{G}^{[r]}$ with $\mathbf{G}^{[r]} \in L^{p/r}(\Omega, L(\Lambda^r\mathbb{E}^3, \Lambda^r L(\mathbb{E}^3)))$. The energy functional is weakly lower semi-continuous by a standard convexity argument for the weakly converging highest order terms ($\mathbf{G}^{[r]},\ r = 1,2,3$) and Fatou's lemma combined with point-wise convergence for the lower order terms (see [13] or for the more general results [4]).

Suppose that a sequence $\{\mathbf{f}_i\} \subset \mathcal{A}^I_{in}$, is converging weakly $\mathbf{f}_i \rightharpoonup \mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3)$. By embedding of $W^{2,p}(\Omega, \mathbb{E}^3)$ in $C^1(\overline{\Omega}, \mathbb{E}^3)$, the linear functional $\mathbf{f}_i \mapsto \nabla \mathbf{f}_i(\mathbf{x})$ is continuous for $\mathbf{x} \in \overline{\Omega}$. Thus $\nabla \mathbf{f}_i(\mathbf{x})$ is converging to $\nabla \mathbf{f}(\mathbf{x})$, and this implies $\det(\nabla \mathbf{f}(\mathbf{x})) = 1$. Similarly, for points in $\partial\Omega$, point-wise convergence implies the limit function satisfies the boundary conditions. $\square$

Remark 2.3. *In fact by an argument in Evans [13], the determinant is weakly continuous from $W^{1,p}(\Omega, \mathbb{E}^n) \to L^{p/n}(\Omega)$ for more delicate reasons. The continuous embedding given by higher differentiability allows for a simpler proof here and other places.*

2.2 Global Injectivity

In this section we only consider the Dirichlet boundary condition problems, **S** and **W**. The result of Lemma 2.1 is strengthened as local invertibility of **f** along with the boundary conditions imply that **f** is globally injective.

Lemma 2.4. *If $\mathbf{f} \in \mathcal{A}^S_{in}$ or $\mathbf{f} \in \mathcal{A}^W_{in}$, then $\mathbf{f}$ is a bijection of $\overline{\Omega}$ and $\mathbf{f}_0(\overline{\Omega})$.*

Proof. By Sobolev embedding, **f** is continuously differentiable on $\overline{\Omega}$. Since $\det(\nabla \mathbf{f}(\mathbf{x})) = 1$ for all $\mathbf{x} \in \overline{\Omega}$, it follows that **f** is a local diffeomorphism.

Consider the topological degree of **f**, i.e. $\deg(\mathbf{f}, \Omega, \mathbf{y})$ for $\mathbf{y} \in \mathbf{f}_0(\Omega)$ [11]. Then since $\mathbf{f}$ and $\mathbf{f}_0$ agree on $\partial\Omega$ and $\mathbf{f}_0^{-1}(\mathbf{y}) \cap \partial\Omega = \emptyset$, homotopy invariance implies $\deg(\mathbf{f}, \Omega, \mathbf{y}) = \deg(\mathbf{f}_0, \Omega, \mathbf{y}) = 1$ for all $\mathbf{y} \in \mathbf{f}_0(\Omega)$. Since $\mathbf{f} \in C^1(\overline{\Omega}, \mathbb{E}^3)$ and $\nabla\mathbf{f}$ is everywhere non-singular, for any $\mathbf{y} \in \mathbf{f}_0(\Omega)$ the degree is computed as

$$\deg(\mathbf{f}, \Omega, \mathbf{y}) = \sum_{\mathbf{x}\in\mathbf{f}^{-1}(\mathbf{y})} \operatorname{sign}\, \det(\nabla\mathbf{f}(\mathbf{x})). \tag{2.7}$$

The sign of $\det(\nabla\mathbf{f}(\mathbf{x}))$ is always 1, and because the degree is 1 this implies that there is exactly one $\mathbf{x}$ in the pre-image of $\mathbf{y}$. For $\mathbf{y} \notin \mathbf{f}_0(\overline{\Omega})$ the degree is zero and (2.7) implies that $\mathbf{f}^{-1}(\mathbf{y}) = \emptyset$. It remains to show that for each $\mathbf{y} \in \mathbf{f}_0(\partial\Omega)$ there is exactly one $\mathbf{x} \in \mathbf{f}^{-1}(\partial\Omega)$. There is at least one such $\mathbf{x}$ as $\mathbf{f}$ and $\mathbf{f}_0$ agree on $\partial\Omega$ and $\mathbf{f}_0$ is a bijection. If there were more than one $\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{y})$, then there would be $\mathbf{x}_1 \in \mathbf{f}^{-1}(\mathbf{y}) \cap \Omega$, but since $\mathbf{f}$ is open, i.e. locally $\mathbf{f}^{-1}$ is continuous, this implies there is some $\mathbf{x}_2 \in \Omega$ near $\mathbf{x}_1$ that maps outside of $\mathbf{f}_0(\overline{\Omega})$, which is a contradiction. □

Remark 2.4. *An analogous result holds in the context of classical, first-gradient, elasticity [3]. In that case we work with* $\mathbf{f} \in W^{1,p}(\Omega, \mathbb{E}^3)$ *with* $\det(\nabla\mathbf{f}(\mathbf{x})) > 0$ *for almost every* $\mathbf{x} \in \Omega$ *and the result is that* $\mathbf{f}$ *is almost everywhere invertible. We require the stronger result, and again the stronger assumptions admit a simpler proof.*

2.3 Euler-Lagrange Equations

Our next claim is that $E : W^{2,p}(\Omega, \mathbb{E}^3) \to \mathbb{R}$ and $H : W^{2,p}(\Omega, \mathbb{E}^3) \to W^{1,p}(\Omega)$ are continuously Fréchet differentiable (see Definition A.2 in Appendix A). Let $\mathbf{F} = \nabla\mathbf{f}$ and $\mathbf{G} = \nabla^2\mathbf{f}$ then, repressing explicit spatial dependence of $\mathbf{v}, \mathbf{F}$ and $\mathbf{G}$ in the integrands, the derivatives are given by

$$
\begin{aligned}
DE[\mathbf{f}]\mathbf{v} = &\int_\Omega \left[\nabla\mathbf{v}\cdot D_F W(\mathbf{F},\mathbf{G},\mathbf{x}) + \nabla^2\mathbf{v}\cdot D_G W(\mathbf{F},\mathbf{G},\mathbf{x}) + \mathbf{v}\cdot D_f g(\mathbf{f},\mathbf{x})\right] dV \\
&+ \int_{\partial\Omega} \mathbf{v}\cdot D_f\gamma(\mathbf{f},\mathbf{x}) dS, \\
(DH[\mathbf{f}]\mathbf{v})(\mathbf{x}) = &\nabla\mathbf{v}(\mathbf{x})\cdot \operatorname{cof}(\mathbf{F}(\mathbf{x})).
\end{aligned} \tag{2.8}
$$

2.3.1 Fréchet Differentiability

To prove continuous Fréchet differentiability, we verify the hypothesis of the three cases of Theorem A.1, which relate to

1. maps from $L^p(\Omega) \to L^r(\Omega)$ with $p > r$,
2. maps from $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ with $p > 3$,
3. and maps from $C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in [0, 1]$.

From **A**2, $\mathbf{G} \mapsto W$ is C^1, and satisfies the growth conditions $|W(\mathbf{F},\mathbf{G},\mathbf{x})| \leq C_{G1}(\mathbf{F})(|\mathbf{G}|^p + 1)$ and $|D_G W(\mathbf{F},\mathbf{G},\mathbf{x})| \leq C_{G2}(\mathbf{F})(|\mathbf{G}|^{p-1} + 1)$, meeting the hypothesis for case 1.

To show continuous differentiability of H it is sufficient that $\mathbf{F} \mapsto \det(\mathbf{F})$ is C^2, i.e. case 2. For $\mathbf{F} \mapsto W$, $\mathbf{f} \mapsto g$, and $\mathbf{f} \mapsto \gamma$, assumptions **A**2 and **A**3 imply that the operators are continuously differentiable from $C^0(\overline{\Omega}, L(\mathbb{E}^3))$ (or $C^0(\overline{\Omega}, \mathbb{E}^3)$) to $C^0(\overline{\Omega})$ (or $C^0(\partial\Omega)$), as covered by case 3.

Combining the result of Theorem A.1 with Fréchet differentiability of the bounded linear operators of integration, $W \mapsto \int_\Omega W\ dV$, differentiation, e.g. $\mathbf{f} \mapsto \nabla^2\mathbf{f}$, and Sobolev embedding proves continuous Fréchet differentiability of E and H.

2.3.2 Divergence Term Surjectivity

To prove existence of the pressure we use the Lagrange multiplier theorem in the form given in Appendex B, Theorem B.5. Along with continuous differentiability, the other significant hypothesis of Theorem B.5 is that the linearized constraint, (2.8), has closed range. The linearized constraint has the form of a divergence operator and is either surjective or has a one dimensional co-kernel depending on the boundary conditions. The equation, $\nabla\mathbf{v} \cdot \mathrm{cof}(\mathbf{F}) = h$, has many solutions, but to prove surjectivity we must select one with consistent boundary conditions and enough regularity so that the solution lies in $W^{2,p}(\Omega, \mathbb{E}^3)$.

For the strong Dirichlet boundary value problem, $\mathbf{S}$, the main technical result can be found in Galdi's book on the Navier-Stokes equations [14], where he shows that for Lipschitz domains the divergence operator has a one-dimensional co-kernel from $W_0^{2,p}(\Omega, \mathbb{E}^3) \to W_0^{1,p}(\Omega)$ corresponding to the necessary restriction that $\int_\Omega h\ dV = 0$. This result was first published by Bogovskii in 1979 [8] and we state a version in Lemma 4.2 of Chapter 4. Combined with the result showing admissible deformations are globally invertible, and continuous differentiability of the composition operator on the space $W_0^{2,p}(\Omega, \mathbb{E}^3)$, changing coordinates to the current configuration proves the needed result for the linearized operator.

The weak Dirichlet case requires some regularity results similar to what can be found in [17] and [19], where $W^{2,p}$ regularity is shown for solutions to the Stokes equations with Dirichlet boundary conditions. These results do not follow from the general elliptic estimates of [1] because the strong form estimates would require that $\operatorname{cof}(\mathbf{F})$ is continuously differentiable (it is continuous and weakly differentiable). One could also consider the first order system corresponding to the Hodge decomposition, which requires less regularity for the coefficients than the general elliptic estimates. However, we find that only the components of the vector field tangential to the boundary may be prescribed by this method. The method of Ladyzhenskaya [25] (sketched in Lemma 4.1) to correct for the normal component of the vector field does not achieve the optimal regularity that we need. A weaker form of the Schauder estimates, however, has been shown to be valid for Stokes-like systems in [17], and these estimates are sufficient to prove higher regularity. The assumptions in [17] and [19] are of class C^2, but in Chapter 4 we analyze the proof and it is sufficient for the boundary to be $W^{2,p}$. It would not grant us any convenience to assume that $\partial\Omega$ and $\mathbf{f}_0$ are C^2, because the change of coordinates using $\mathbf{f}$ would lose the additional regularity.

The mixed boundary conditions present two additional challenges. The possible loss of regularity at the transition between Dirichlet and Neumann boundary conditions means that it is not sufficient to solve Stokes equations with mixed boundary conditions. There is also the possibility of loss of global invertibility for the deformation and thus we cannot globally change coordinates into the current configuration to solve a divergence equation. A combination of the methods used in the **W** and **S** cases allows us to address these issues.

We define a subspace of $W_0^{1,p}(\Omega)$ by

$$W_0^{1,p}(\Omega)/\mathbb{R} \equiv \left\{ h \in W_0^{1,p}(\Omega) : \int_\Omega h \, dV = 0 \right\}.$$

Lemma 2.5. *For* $\mathbf{f} \in \mathcal{A}_{in}^S$, *the linearized constraint (see (2.8))* $DH[\mathbf{f}] : W_0^{2,p}(\Omega, \mathbb{E}^3) \to W_0^{1,p}(\Omega)/\mathbb{R}$ *is surjective.*

Proof. Let $\mathbf{F} = \nabla \mathbf{f}$ and consider the equation

$$\nabla \mathbf{v} \cdot \operatorname{cof}(\mathbf{F}) = h \tag{2.9}$$

for $h \in W_0^{1,p}(\Omega)$ such that $\int_\Omega h \, dV = 0$. It must be shown that there exists a solution $\mathbf{v} \in W_0^{2,p}(\Omega, \mathbb{E}^3)$. The result of Lemma 2.4 shows that $\mathbf{f}$ is globally invertible, as well as locally volume and orientation preserving. We change coordinates to the current configuration by letting $\mathbf{w}(\mathbf{y}) = \mathbf{v}(\mathbf{f}^{-1}(\mathbf{y}))$ for $\mathbf{y} \in \mathbf{f}(\Omega)$. The chain rule, along with that $\operatorname{cof}(\mathbf{F}) = \mathbf{F}^{-\top} = (\nabla_y \mathbf{f}^{-1})^\top$, shows that

$$\begin{aligned} \nabla_x \mathbf{v}(\mathbf{x}) \cdot \operatorname{cof}(\mathbf{F}(\mathbf{x})) =& \nabla_x \mathbf{w}(\mathbf{f}(\mathbf{x})) \cdot \operatorname{cof}(\mathbf{F}(\mathbf{x})) \\ =& \operatorname{tr}\left(\nabla_y \mathbf{w}(\mathbf{y}) \mathbf{F}(\mathbf{x}) \mathbf{F}^{-1}(\mathbf{x}) \right) \\ =& \nabla_y \cdot \mathbf{w}(\mathbf{y}). \end{aligned}$$

We must then solve the divergence equation on the domain $\mathbf{f}(\Omega)$. We find that $g(\mathbf{y}) \equiv h(\mathbf{f}^{-1}(\mathbf{y}))$ satisfies $g \in W_0^{1,p}(\mathbf{f}(\Omega))$, and

$$\int_{\mathbf{f}(\Omega)} g(\mathbf{y}) dV_y = \int_{\Omega} h(\mathbf{x}) \det(\mathbf{F}(\mathbf{x})) dV_x = 0. \tag{2.10}$$

Since $\mathbf{f} \in C^1(\overline{\Omega}, \mathbb{E}^3)$, and $\partial\Omega$ is Lipschitz, $\partial\mathbf{f}(\Omega)$ is Lipschitz and also $\mathbf{f}(\Omega)$ is still connected. The result of Galdi, Theorem 3.2 in [14] (see also Lemma 4.2 in Chapter 4 for an essential step), implies the existence of $\mathbf{w} \in W_0^{2,p}(\mathbf{f}(\Omega), \mathbb{E}^3)$ satisfying

$$\nabla_y \cdot \mathbf{w}(\mathbf{y}) = g(\mathbf{y}).$$

Changing coordinates back to the reference configuration, we get $\mathbf{w}(\mathbf{f}(\mathbf{x})) = \mathbf{v}(\mathbf{x}) \in W_0^{2,p}(\Omega, \mathbb{E}^3)$ and $\mathbf{v}$ satisfies equation (2.9), proving the lemma. □

We define the subspace of displacements that vanish on Γ as

$$W_\Gamma^{2,p}(\Omega, \mathbb{E}^3) = \left\{ \mathbf{u} \in W^{2,p}(\Omega, \mathbb{E}^3) : \ \mathbf{u}(\mathbf{x}) = \mathbf{0} \ \forall \ \mathbf{x} \in \Gamma \right\} \tag{2.11}$$

Lemma 2.6. *For $\mathbf{f} \in \mathcal{A}_{in}^I$ and $\mathbf{I} \in \{\mathbf{W}, \mathbf{M}\}$, and the respective assumptions on the boundary from section 2.0.2, the linearized constraint $DH[\mathbf{f}] : W_\Gamma^{2,p}(\Omega, \mathbb{E}^3) \to W^{1,p}(\Omega)$ has closed range. If $\mathbf{I} = \mathbf{M}$, then $DH[\mathbf{f}]$ is surjective, and if $\mathbf{I} = \mathbf{W}$ then it has a one dimensional co-kernel corresponding to $\int_\Omega h \, dV = 0$.*

Proof. We prove the case of $\mathbf{M}$ in four steps. Step 3 implies the case of global weak Dirichlet boundary conditions, $\mathbf{W}$.

Step 1 We extend $\mathbf{f}$ and h to $\tilde{\mathbf{f}} \in W^{2,p}(\tilde{\Omega}, \mathbb{E}^3)$ and $\tilde{h} \in W_0^{1,p}(\tilde{\Omega}, \mathbb{E}^3)$ for an open, bounded and connected neighborhood $\tilde{\Omega}$ with Lipschitz boundary such that

- $\overline{\Omega} \subset \tilde{\Omega}$,
- $\det(\nabla\tilde{\mathbf{f}}(\mathbf{x})) \geq \delta > 0$ for all $\mathbf{x} \in \tilde{\Omega}$,
- and $\int_{\tilde{\Omega}} \tilde{h}\, dV = 0$.

This is possible by the Sobolev extension theorem because $\partial\Omega$ is assumed to be Lipschitz. The uniform lower bound on $\det(\nabla\tilde{\mathbf{f}})$ follows from the Sobolev embedding into uniformly continuously differentiable functions and continuity of the determinant. It is easy to adjust $\tilde{h}$ to integrate to zero by a smooth bump function supported in $\tilde{\Omega}\backslash\overline{\Omega}$.

Step 2 Then we extend Γ to $\partial\Upsilon$ such that

- Υ is open and connected,
- $\Gamma \subset \partial\Upsilon$ and $\partial\Omega$ is $W^{2,p}$,
- and $\overline{\Upsilon} \subset \tilde{\Omega}$.

The procedure to smooth the boundary while maintaining Γ is shown in Lemma C.4. Furthermore, we select $\mathbf{u} \in W^{2,p}_{\Gamma}(\Upsilon, \mathbb{E}^3)$, such that for $\boldsymbol{\nu}(\mathbf{x})$ the outward pointing unit normal vector at $\mathbf{x} \in \partial\Upsilon$,

$$\int_{\partial\Upsilon} \operatorname{cof}(\nabla\tilde{\mathbf{f}}(\mathbf{x}))\mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, dS = \int_{\Upsilon} \tilde{h}(\mathbf{x}) dV, \tag{2.12}$$

as it is easy to adjust $\mathbf{u}$ in the neighborhood of a point in $\partial\Upsilon\backslash\Gamma$.

Step 3 We consider the equation

$$\operatorname{cof}(\nabla\tilde{\mathbf{f}}(\mathbf{x})) \cdot \nabla\mathbf{v}(\mathbf{x}) = \tilde{h}(\mathbf{x}), \tag{2.13}$$

and solve (2.13) for $\mathbf{v}^0 \in W^{2,p}(\Upsilon, \mathbb{E}^3)$ with $\mathbf{v}^0 - \mathbf{u} \in H_0^1(\Upsilon, \mathbb{E}^3)$ using the solution to a Stokes-like system, proven to exist in Theorem 4.1. Equation (2.12) shows that the right-hand side of (2.13) satisfies the compatibility condition required for the existence of a solution. In the case of $\mathbf{W}$, this is satisfied by $\Upsilon = \Omega$ and $\mathbf{u} = \mathbf{0}$. For the $\mathbf{M}$ case, we then extend $\mathbf{v}^0$ to $\tilde{\mathbf{v}}^0 \in W_0^{2,p}(\tilde{\Omega}, \mathbb{E}^3)$. We set

$$\tilde{h}^0(\mathbf{x}) = \tilde{h}(\mathbf{x}) - \operatorname{cof}(\nabla\tilde{\mathbf{f}}(\mathbf{x})) \cdot \nabla\mathbf{v}^0(\mathbf{x}).$$

Since $\tilde{h}^0(\mathbf{x}) \in W_0^{1,p}(\tilde{\Omega})$, and $\tilde{h}^0$ vanishes for $\mathbf{x} \in \overline{\Upsilon}$, we get that $\tilde{h}^0 \in W_0^{1,p}(\tilde{\Omega}\backslash\overline{\Upsilon})$. Because $\nabla\mathbf{v}^0$ vanishes near the boundary of $\tilde{\Omega}$ and $\nabla \cdot \operatorname{cof}(\nabla\tilde{\mathbf{f}}(\mathbf{x})) = 0$, integration by parts shows that $\int_{\tilde{\Omega}} \tilde{h}^0 \, dV = 0$.

Step 4 Finally, we solve for $\mathbf{v}^1 \in W_0^{2,p}(\tilde{\Omega}\backslash\overline{\Gamma}, \mathbb{E}^3)$ such that

$$\operatorname{cof}(\nabla\tilde{\mathbf{f}}(\mathbf{x})) \cdot \nabla\mathbf{v}^1(\mathbf{x}) = \tilde{h}^0(\mathbf{x}). \tag{2.14}$$

Existence of a solution is shown in Corollary 4.1. Now we let $\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^1$, and $\mathbf{v} \in W^{2,p}(\Omega, \mathbb{E}^3)$, $\mathbf{v}$ vanishes on Γ, and $\mathbf{v}$ solves $DH[\mathbf{f}]\mathbf{v} = h$.

□

2.3.3 Lagrange Multiplier

We have now shown the hypotheses of Theorem B.5. In particular, that the total energy, E, and incompressibility constraint, H, are continuously Fréchet differentiable, and that $DH[\mathbf{f}]$ is surjective with the appropriate co-domain. Thus applying Theorem B.5 proves the following theorem.

Theorem 2.1. *Suppose E satisfies the assumptions of Section 2.0.2, and $\mathbf{f}^*$ is an energy minimizer in the class $\mathcal{A}^I_{in}$ for $\mathbf{I} \in \{\mathbf{S}, \mathbf{W}, \mathbf{M}\}$. Then there exists a pressure, p, such that*

$$0 = \langle DE[\mathbf{f}^*], \mathbf{v}\rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} - \langle p, DH[\mathbf{f}^*]\mathbf{v}\rangle_{W^{1,p}(\Omega)}. \tag{2.15}$$

- *In the case $\mathbf{I} = \mathbf{S}$, (2.15) holds for all $\mathbf{v} \in W_0^{2,p}(\Omega, \mathbb{E}^3)$ and p is in*
$$\left(W_0^{1,p}(\Omega)/\mathbb{R}\right)^* = \left\{p \in W_0^{1,p}(\Omega)^* : \langle p, 1\rangle_{W_0^{1,p}(\Omega)} = 0\right\}.$$
- *For $\mathbf{I} = \mathbf{W}$, (2.15) holds for $\mathbf{v}$ that vanish on $\partial\Omega$ and similarly $p \in (W^{1,p}(\Omega)/\mathbb{R})^*$.*
- *If $\mathbf{I} = \mathbf{M}$, (2.15) holds for $\mathbf{v}$ that vanish on Γ and $p \in W^{1,p}(\Omega)^*$.*

The variational equation involves a distributional pressure, in some cases defined up to a constant. To more explicitly characterize the pressure for the strong Dirichlet case, $\mathbf{S}$, we can express the equations in terms of $r \in W_0^{1,p/(p-1)}(\Omega)$ by solving ($-\Delta r = p$ with Dirichlet boundary conditions)

$$\int_\Omega \nabla r \cdot \nabla q \, dV = \langle p, q\rangle_{W_0^{1,p}(\Omega)} \ \forall\, q \in W_0^{1,p}(\Omega).$$

Existence of such an r follows from the invertibility of the Laplacian as an operator $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, and the regularity of solutions to the weak Laplace equation. In terms of $r \in W_0^{1,p/(p-1)}(\Omega)$ and $\mathbf{F} = \nabla\mathbf{f}$ and $\mathbf{G} = \nabla^2\mathbf{f}$, the weak form of the equations appear as

$$\int_\Omega \left[\nabla\mathbf{v} \cdot D_F W(\mathbf{F}, \mathbf{G}, \mathbf{x}) + \nabla^2\mathbf{v} \cdot D_G W(\mathbf{F}, \mathbf{G}, \mathbf{x}) - \nabla r \cdot \nabla(\nabla\mathbf{v} \cdot \mathrm{cof}(\mathbf{F}))\right] dV = 0 \tag{2.16}$$

for all $\mathbf{v} \in W_0^{2,p}(\Omega, \mathbb{E}^3)$. Equivalently, r may be considered as the Lagrange multiplier of the weakly imposed incompressibility constraint:

$$\int_\Omega \nabla q \cdot \nabla \det(\nabla \mathbf{f}) dV = 0 \tag{2.17}$$

for all $q \in W_0^{1,p/(p-1)}(\Omega)$ combined with $\int_\Omega \det(\nabla \mathbf{f})\, dV = |\Omega|$.

2.4 Example

This example is sometimes called Gurtin's experiment in [11] and has roots in the works of Truesdell and Rivlin.

We consider a problem posed on a solid cylinder, $\Omega = D^2 \times [0, 1]$. Let the stored energy satisfy the **M** assumptions for the mixed boundary conditions and let $\mathbf{f}_0$ correspond to an isometric embedding into $\mathbb{E}^3$, $\mathbf{f}_0(\mathbf{x}) = \mathbf{x}$, where we have chosen the coordinates so that the cylinder is described by $\mathbf{x} \cdot \mathbf{e}_3 \in [0, 1]$ and $(\mathbf{x} \cdot \mathbf{e}_1)^2 + (\mathbf{x} \cdot \mathbf{e}_2)^2 \leq 1$.

We impose Dirichlet boundary conditions at the ends of the cylinder, explicitly that $\mathbf{f}(\mathbf{x}) - \mathbf{f}_0(\mathbf{x}) = \mathbf{0}$ whenever $\mathbf{x} \cdot \mathbf{e}_3 \in \{0, 1\}$. On the rest of the domain we use natural boundary conditions.

Proposition 2.2. *Suppose the total energy function $E[\mathbf{f}]$ satisfies the assumptions of Section 2.0.2. Then there exist at least two solutions in $\mathcal{A}_{in}^M$ to the weak equilibrium equations (2.15) for the problem posed above.*

Proof. Proposition 2.1 implies there is a global energy minimizer, $\mathbf{f}^* \in \mathcal{A}_{in}^M$, and

Theorem 2.1 implies that $\mathbf{f}^*$ satisfies (2.15).

A second solution comes from restricting to a different homotopy class of maps. For this problem, the class of admissible displacements, $\mathcal{A}_{in}^M$, has at least two connected components, both of which are weakly closed. We distinguish two components by considering the deformation gradient restricted to the center line, let $t = \mathbf{x} \cdot \mathbf{e}_3$ and $\mathbf{F}_t = \nabla\mathbf{f}(t\mathbf{e}_3)$ for $\mathbf{f} \in \mathcal{A}_{in}^M$. From the Dirichlet boundary conditions on the ends of the cylinder, for $t \in \{0, 1\}$, $\mathbf{F}_t\mathbf{x} = \mathbf{x}$ whenever $\mathbf{x} \cdot \mathbf{e}_3 = 0$, and $\mathbf{e}_3 \cdot \mathbf{F}_t\mathbf{e}_3 = 1$ due to the incompressibility constraint. Thus the only difference is a shear, so we define $\mathbf{F} : (1, 2] \to L(\mathbb{E}^3)$ by

$$\mathbf{F}_t = \mathbf{F}_1 - (t - 1)(\mathbf{F}_1\mathbf{e}_3) \otimes \mathbf{e}_3,$$

and $\mathbf{F} : [-1, 0) \to L(\mathbb{E}^3)$ by

$$\mathbf{F}_t = \mathbf{F}_0 + t(\mathbf{F}_0\mathbf{e}_3) \otimes \mathbf{e}_3.$$

Then $\mathbf{F} : [-1, 2] \to SL(3)$ and $\mathbf{F}_0 = \mathbf{F}_2$ is the identity. We identify to $\mathbf{f}$ the class of this map in the fundamental group, $\pi_1(SL(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Clearly, $\mathbf{f}_0$ represents the trivial class, whereas the non-trivial element is represented by $\mathbf{f}_1(\mathbf{x}) = \mathbf{R}_{2\pi t}\mathbf{x}$, for $\mathbf{R}_{2\pi t}$ a rotation about the $\mathbf{e}_3$ axis by $2\pi t$, because

$$\nabla\mathbf{f}_1(\mathbf{x})\mathbf{e}_i = \mathbf{R}_{2\pi t}\mathbf{e}_i + 2\pi(\mathbf{e}_3 \times \mathbf{x})(\mathbf{e}_3 \cdot \mathbf{e}_i)$$

so $\nabla\mathbf{f}_1(t\mathbf{e}_3) = \mathbf{R}_{2\pi t}$. The global minimizer may belong to either class.

We consider the problem of restricting the admissible deformations to maps that represent the other homotopy class than $\mathbf{f}^*$. We claim this set is weakly closed thus there exists an energy minimizer $\mathbf{f}_1^*$. Suppose $\mathbf{f}^i \rightharpoonup \mathbf{f}$ then $\mathbf{F}^i \to \mathbf{F}$ in $C([0,1], L(\mathbb{E}^3))$, and it follows that $\mathbf{F}$ represents the same homotopy class.

Finally, we show that the set of restricted deformations is relatively open with respect to the strong topology on $W^{2,p}(\Omega, \mathbb{E}^3)$, and thus $\mathbf{f}_1^*$ is a local energy minimizer in $\mathcal{A}_{in}^M$, and Theorem 2.1 applies to show that $\mathbf{f}_1^*$ satisfies (2.15). This follows from the embedding into continuously differentiable functions and that the homotopy classes are relatively open in $C([0,1], L(\mathbb{E}^3))$. $\square$

2.5 Conclusion

In this chapter we pose a set of reasonable assumptions for second-gradient elasticity, and show the existence of an incompressible equilibrium solution for large deformations. We combine the analysis of existence of a global incompressible minimum, differentiability of the energy and constraint, and surjectivity of the linearized incompressibility constraint operator under the unified set of assumptions. The second-gradient assumptions provide the needed regularity to combine the above tools in a consistent matter to apply the Lagrange multiplier theorem for Banach spaces.

For the first-gradient model, existence of minimizers has been shown in [5] for the class of $W^{1,p}(\Omega, \mathbb{E}^3)$ deformations with $p > 3$. From the differentiability theory of Appendix A it is not difficult to show the energy is continuously Fréchet differentiable and the constraint is continuously Fréchet differentiable mapping $H : W^{1,p}(\Omega, \mathbb{E}^3) \to L^{p/3}(\Omega)$. As mentioned in Remark 2.3, H is weakly continuous on these spaces. This is, however, insufficient to provide equilibrium solutions because $DH[\mathbf{f}] : W^{1,p}(\Omega, \mathbb{E}^3) \to L^{p/3}(\Omega)$ fails to be surjective. For example, if $\mathbf{f}$ is smooth then it is easy to see that the range of the linearized constraint is contained in $L^p(\Omega)$, which is a strict subset of $L^{p/3}(\Omega)$. On the other hand, while $DH[\mathbf{f}] : W^{1,p}(\Omega, \mathbb{E}^3) \to L^p(\Omega)$ is surjective under certain assumptions, the non-linear map does not map $W^{1,p}(\Omega, \mathbb{E}^3)$ to $L^p(\Omega)$ and certainly is not continuous or differentiable.

Even in the second-gradient case, surjectivity of the linearized determinant operator is the most technical part of the analysis. The ability to handle mixed boundary conditions is a highlight of our results. Allowing for corner singularities at the interface of boundary conditions includes many physically relevant scenarios.

A short-coming of our approach is the lack of higher regularity. Higher regularity of an equilibrium solution would tie together global minimizers with the local theory built on linearization of the equilibrium equations for the cases of Dirichlet boundary conditions. Higher regularity would also allow statements about the boundedness of the pressure and stresses in the material. The contribution of the second-gradient to the total energy would be limited if we knew the second derivatives do not get to large. Uniform estimates would allow a rigorous limit to a first-gradient model. In any case, showing the minimizers are equilibrium is a first step to regularity analysis. We find it likely that results on partial regularity will apply to the equilibrium of (2.15), although we are not aware of any results that imply partial regularity in the generality needed for our higher/mixed order non-linear systems. Even with the most generous choices for the stored energy function, i.e. W is smooth and uniformly striclty convex in $\nabla^2\mathbf{f}$, we do not believe the tools have been developed to show full regularity.

CHAPTER 3

SELF-CONTACT AND GLOBAL INJECTIVITY OF WEAK SOLUTIONS IN MIXED BOUNDARY-VALUE PROBLEMS

3.1 Notation and Problem Formulation

We consider an extension of problem **M** of Chapter 2 to find globally injective equilibria, which we refer to by the superscript **C**. To formulate the new constraint, we define the admissible set as

$$\begin{aligned}\mathcal{A}^C = \Big\{ &\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3) : \mathbf{f}(\mathbf{x}) = \mathbf{f}_0(\mathbf{x}) \; \forall\, \mathbf{x} \in \Gamma, \\ &\det(\nabla \mathbf{f}(\mathbf{x})) > 0 \; \forall\, \mathbf{x} \in \overline{\Omega},\; \mathbf{f} \text{ is injective on } \Omega \Big\}. \end{aligned} \tag{3.1}$$

Incompressibility is imposed via the constraint $\det(\nabla \mathbf{f}(\mathbf{x})) = 1$ for $\mathbf{x} \in \overline{\Omega}$. In this case the admissible incompressible deformations form a subset of $\mathcal{A}^C$ given by

$$\begin{aligned}\mathcal{A}^C_{in} = \Big\{ &\mathbf{f} \in W^{2,p}(\Omega, \mathbb{E}^3) : \mathbf{f}(\mathbf{x}) = \mathbf{f}_0(\mathbf{x}) \; \forall\, \mathbf{x} \in \Gamma, \\ &\det(\nabla \mathbf{f}(\mathbf{x})) = 1 \; \forall\, \mathbf{x} \in \overline{\Omega},\; \mathbf{f} \text{ is injective on } \Omega \Big\}. \end{aligned} \tag{3.2}$$

Let $E[\mathbf{f}]$ be a total energy functional as in (2.2). The compressible energy minimization problem is to minimize $E[\mathbf{f}]$ subject to $\mathbf{f} \in \mathcal{A}^C$ and the incompressible version instead restricts to $\mathbf{f} \in \mathcal{A}^C_{in}$. In some cases we use the notation $\mathcal{A}^C_{(in)}$ to refer to either the compressible or incompressible case.

We denote the tangent space at an admissible deformation $\mathbf{f} \in \mathcal{A}^C$ to be

$$T_f\mathcal{A}^C = \left\{\mathbf{h} \in W^{2,p}(\Omega, \mathbb{E}^3) : \mathbf{h}(\mathbf{x}) = \mathbf{0} \; \forall \; \mathbf{x} \in \Gamma\right\}, \tag{3.3}$$

or in the incompressible case, for $\mathbf{f} \in \mathcal{A}^C_{(in)}$,

$$T_f\mathcal{A}^C_{in} = \left\{\mathbf{h} \in W^{2,p}(\Omega, \mathbb{E}^3) : \mathbf{h}(\mathbf{x}) = \mathbf{0} \; \forall \; \mathbf{x} \in \Gamma, \; \mathrm{cof}(\nabla\mathbf{f}(\mathbf{x})) \cdot \nabla\mathbf{h}(\mathbf{x}) = 0 \; \forall \; \mathbf{x} \in \overline{\Omega}\right\}. \tag{3.4}$$

We define the interior cone for $\mathbf{f} \in \mathcal{A}^C_{(in)}$:

$$\begin{aligned} K_f\mathcal{A}^C_{(in)} = \Big\{\mathbf{h} \in T_f\mathcal{A}^C_{(in)} : \exists \; \epsilon_1 > 0 \\ \text{and } \mathbf{v} \in C([0, \epsilon_1), \mathcal{A}^C_{(in)}) \; s.t. \; \mathbf{v}_0 = \mathbf{f} \text{ and } \mathbf{v}_\epsilon = \mathbf{f} + \epsilon\mathbf{h} + o(\epsilon)\Big\}, \end{aligned} \tag{3.5}$$

which is equivalent to a derivable tangent cone defined in [34].

3.2 Assumptions

We assume that $\partial\Omega$ is Lipschitz, which is used for a variety of properties of the domain, including the Sobolev extension property, and the existence of non-empty interior cones.

We also assume that the energy is coercive, i.e.,

$$\|\mathbf{f}\|_{W^{2,p}(\Omega,\mathbb{E}^3)} \leq C_c(E[\mathbf{f}] + 1), \tag{3.6}$$

continuously Fréchet differentiable and weakly lower semi-continuous. See Section 2.0.2 and 2.1 for the assumptions on W that how they imply the differentiability and semi-continuity properties. We make the assumptions on $\partial\Omega$ of case **M** in Section 2.0.2. In particular, we assume that $\mathbf{f}_0$ is injective on $\overline{\Omega}$.

In the compressible case, we also assume the following condition:

$$\sup_{\mathbf{x}\in\Omega} \left|\det\left(\nabla\mathbf{f}(\mathbf{x})\right)\right|^{-1} \leq C_d(E[\mathbf{f}], \mathbf{f}_0, \Omega, \Gamma), \ \forall\, \mathbf{f} \in \mathcal{A}^C. \tag{3.7}$$

See [23] for the conditions to determine when this property is satisfied.

For the incompressible case, a necessary property is summarized as a corollary of Lemma 2.6.

Corollary 3.1. *Suppose* $\mathbf{f} \in \mathcal{A}^C_{in}$ *and* $\mathbf{h} \in T_f\mathcal{A}^C_{in}$*. Then there is* $\epsilon_1 > 0$ *and a continuous trajectory* $\mathbf{v} : [0, \epsilon_1) \to \mathcal{A}^M_{in}$ *such that* $\mathbf{v}(0) = \mathbf{f}$ *and* $\mathbf{v}$ *is differentiable at* 0 *with* $\frac{d}{d\epsilon}\mathbf{v}(0) = \mathbf{h}$*. Or with the notation of (3.4) and (3.5) for* $\mathcal{A}^M_{in}$*, simply* $K_f\mathcal{A}^M_{in} = T_f\mathcal{A}^M_{in}$*.*

Proof. The incompressibility constraint operator, $H : \mathcal{A}^M_{in} \to W^{1,p}(\Omega)$, is continuously differentiable as a consequence of Theorem A.1 following the argument in Section 2.3.1. Then in Lemma 2.6 we show that the linearized operator, $DH[\mathbf{f}]\mathbf{h}(\mathbf{x}) = \operatorname{cof}(\nabla\mathbf{f}(\mathbf{x})) \cdot \mathbf{h}(\mathbf{x})$ is surjective. Thus given $\mathbf{h} \in \ker(DH[\mathbf{f}])$ we use the surjective implicit function theorem, Theorem B.4, to show the existence of the trajectory $\mathbf{v}$. □

3.3 Existence of Globally Injective Minimizers

The existence theory is similar to Section 2.1 and uses the direct method of the calculus-of-variations. Here we demonstrate that global injectivity is preserved under weak convergence of an infimizing sequence.

Proposition 3.1. *Given the assumptions of Section 3.2, there exists* $\mathbf{f}^* \in \mathcal{A}^C_{(in)}$ *such that* $E[\mathbf{f}^*] \leq E[\mathbf{f}]$ *for all* $\mathbf{f} \in \mathcal{A}^C_{(in)}$.

Proof. Suppose $\{\mathbf{f}_i\}_{i=1}^{\infty} \subset \mathcal{A}^C_{(in)}$ is an energy infimizing sequence. By coercivity of E, Lemma 2.2, there is a constant M such that $\|\mathbf{f}_i\|_{W^{2,p}(\Omega,\mathbb{E}^3)} \leq M$. Thus due to the Banach-Alaoglu theorem, there is a subsequence such that $\mathbf{f}_{i_k} \rightharpoonup \mathbf{f}^* \in W^{2,p}(\Omega, \mathbb{E}^3)$. By weak lower semi-continuity of E, which is shown in Lemma 2.3, $E[\mathbf{f}^*] \leq \liminf_{k\to\infty} E[\mathbf{f}_{i_k}]$ so $\mathbf{f}^*$ achieves the infimum and $E[\mathbf{f}^*] \leq E[\mathbf{f}]$ for all $\mathbf{f} \in \mathcal{A}^C_{(in)}$.

For the incompressible case, Lemma 2.3 also shows that the constraint is weakly continuous, hence $\det(\nabla\mathbf{f}^*(\mathbf{x})) = 1$ for all $\mathbf{x} \in \Omega$ and $\mathbf{f}^* \in \mathcal{A}^M_{in}$. In the compressible case, since $\mathbf{f}^*$ has finite energy, assumption (3.7) implies that $\sup_{\mathbf{x}\in\Omega} \det(\nabla\mathbf{f}^*(\mathbf{x}))^{-1} \leq C_d$. In both cases $\mathbf{f}^* \in C^1(\overline{\Omega}, \mathbb{E}^3)$ and is a local diffeomorphism.

It remains to show that $\mathbf{f}^*$ is injective on Ω. We first obtain a uniform bound on the inverse gradient of $\mathbf{f}_{i_k}$. By Cramer's rule

$$\begin{aligned}
\sup_{\mathbf{x}\in\Omega} \left|\nabla\mathbf{f}_{i_k}(\mathbf{x})^{-1}\right| &= \sup_{\mathbf{x}\in\Omega} \left\{\left| \det\left(\nabla\mathbf{f}_{i_k}(\mathbf{x})\right)^{-1}\right| \left|\operatorname{cof}(\nabla\mathbf{f}_{i_k}(\mathbf{x}))\right|\right\} \\
&\leq C \sup_{\mathbf{x}\in\Omega} \left\{\left| \det\left(\nabla\mathbf{f}_{i_k}(\mathbf{x})\right)^{-1}\right|\right\} \sup_{\mathbf{x}\in\Omega} \left\{\left|\nabla\mathbf{f}_{i_k}(\mathbf{x})\right|^2\right\}.
\end{aligned} \tag{3.8}$$

The inverse determinant is either 1 in the incompressible case or bounded by C_d, and the supremum of the gradient of $\mathbf{f}_{i_k}$ is bounded by the $W^{2,p}(\Omega, \mathbb{E}^3)$ norm due to the Sobolev embedding into $C^{1,\alpha}(\overline{\Omega}, \mathbb{E}^3)$. The Sobolev norm is uniformly bounded by an upper bound of the energy due to coercivity.

Suppose that $\mathbf{y} = \mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$ for $\{\mathbf{x}_1, \mathbf{x}_2\} \subset \Omega$. Then there is a closed ball B

and a large integer N such that $\{\mathbf{y}\} \cup_{k=N}^{\infty} \{\mathbf{f}_{i_k}(\mathbf{x}_1), \mathbf{f}_{i_k}(\mathbf{x}_2)\} \subset B \subset \mathbf{f}^*(\Omega) \cup_{k=N}^{\infty} \mathbf{f}_{i_k}(\Omega)$ by uniform convergence of $\mathbf{f}_{i_k}$ and openness of $\mathbf{f}_{i_k}(\Omega)$ and $\mathbf{f}^*(\Omega)$. Then

$$\begin{aligned} |\mathbf{x}_1 - \mathbf{x}_2| &= \left| \int_0^1 \nabla \mathbf{f}_{i_k}^{-1}(t\mathbf{f}_{i_k}(\mathbf{x}_1) + (1-t)\mathbf{f}_{i_k}(\mathbf{x}_2))[\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)]dt \right| \\ &\leq \sup_{\mathbf{x}\in\Omega} \left|\nabla \mathbf{f}_{i_k}^{-1}(\mathbf{x})\right| \left|\mathbf{f}_{i_k}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\right|, \end{aligned} \tag{3.9}$$

which approaches zero because $\lim_{k\to\infty} \mathbf{f}_{i_k}(\mathbf{x}_1) = \lim_{k\to\infty} \mathbf{f}_{i_k}(\mathbf{x}_2) = \mathbf{y}$. Thus $\mathbf{x}_1 = \mathbf{x}_2$ and $\mathbf{f}^*$ is injective on Ω hence $\mathbf{f}^* \in \mathcal{A}_{(in)}^C$. □

3.4 Variations of Energy Minimizers

From the abstract definition of interior cones, (3.5), we easily show that energy minimizers satisfy a variational inequality.

Proposition 3.2. *Suppose $\mathbf{f}^* \in \mathcal{A}_{(in)}^C$ is an energy minimizer, then*

$$\langle DE[\mathbf{f}^*], \mathbf{h}\rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} \geq 0 \;\forall\; \mathbf{h} \in K_f\mathcal{A}_{(in)}^C. \tag{3.10}$$

Proof. From the definition of $K_f\mathcal{A}_{(in)}^C$ there as a continuous map $\mathbf{v} : [0, \epsilon_1) \to \mathcal{A}_{(in)}^C$ such that $\mathbf{v}_0 = \mathbf{f}^*$ and $\frac{d}{d\epsilon}\mathbf{v}_0 = \mathbf{h}$. Since $\mathbf{f}^*$ is an energy minimizer, $E[\mathbf{v}_\epsilon] \geq E[\mathbf{f}^*]$. Then (3.10) follows from the chain rule and continuous Fréchet differentiability of the energy. □

The rest of this section is devoted to characterizing enough of the elements of $K_f\mathcal{A}_{(in)}^C$ to show concrete consequences of the variational inequality, in particular the existence of a measure-valued surface traction.

We first address the somewhat simpler case where we assume that $\partial\Omega$ is C^1. Given $\mathbf{f} \in \mathcal{A}^C$, we define the coincidence set

$$S_f = \left\{ \mathbf{x}_1 \in \overline{\Omega} : \exists\, \mathbf{x}_2 \in \overline{\Omega}\ s.t.\ \mathbf{x}_1 \neq \mathbf{x}_2 \text{ and } \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2) \right\}. \tag{3.11}$$

For $\mathbf{x} \in \partial\Omega$, let $\mathbf{n}_f(\mathbf{x})$ denote the outward pointing unit normal vector of $\mathbf{f}(\overline{\Omega})$ at $\mathbf{f}(\mathbf{x})$.

This lemma is similar to Theorem 5.6-2 in [11].

Lemma 3.1. *Let* $\mathbf{f} \in \mathcal{A}^C$. *Then* S_f *is a closed subset of* $\partial\Omega$, *and for each* $\mathbf{y} \in \mathbf{f}(S_f)$, $\mathbf{f}^{-1}(\mathbf{y}) = \{\mathbf{x}_1, \mathbf{x}_2\}$ *with*

$$\mathbf{n}_f(\mathbf{x}_1) + \mathbf{n}_f(\mathbf{x}_2) = \mathbf{0}. \tag{3.12}$$

Proof. Suppose $\mathbf{x}_1 \in S_f$ and $\mathbf{y} = \mathbf{f}(\mathbf{x}_1)$. By the definition of S_f there exists at least one more point in $\overline{\Omega}$ that maps to $\mathbf{y}$, so call it $\mathbf{x}_2$. Either $\mathbf{x}_1$ or $\mathbf{x}_2$ is in $\partial\Omega$ because $\mathbf{f}$ is injective on Ω, so we assume that $\mathbf{x}_1 \in \partial\Omega$.

We first show that $\mathbf{x}_2 \in \partial\Omega$ and (3.12) holds. In the case that $\mathbf{x}_2 \in \partial\Omega$, suppose for contradiction that $\mathbf{0} \neq \mathbf{z} \equiv \mathbf{n}_f(\mathbf{x}_1) + \mathbf{n}_f(\mathbf{x}_2)$. If $\mathbf{x}_2 \in \Omega$ then $\mathbf{0} \neq \mathbf{z} \equiv \mathbf{n}_f(\mathbf{x}_1)$. We claim that for some small $\epsilon_1 > 0$ there exists two distinct paths of solutions, $\mathbf{x} = \mathbf{w}_\alpha(\epsilon)$, $\alpha \in \{1, 2\}$, to

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \epsilon) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{y} + \epsilon\mathbf{z}, \tag{3.13}$$

with $\mathbf{w}_\alpha \in C([0, \epsilon_1), \mathbb{E}^3)$ such that $\mathbf{w}_\alpha(0) = \mathbf{x}_\alpha$ and $\mathbf{w}_\alpha(\epsilon) \in \Omega$ for $\epsilon \in (0, \epsilon_1)$. This contradicts the injectivity of $\mathbf{f}$ in Ω.

To show the claim, extend $\mathbf{f}$ to a neighborhood of $\overline{\Omega}$ by the Sobolev extension property. Since $\mathbf{f}$ is C^1 and $\det(\nabla \mathbf{f}(\mathbf{x}_\alpha)) > 0$, $\nabla_x \mathbf{g}(\mathbf{x}, 0)$ is invertible, and the implicit function theorem provides paths of solutions to (3.13) in neighborhoods of $\mathbf{x}_1$ and $\mathbf{x}_2$. Then if $\mathbf{x}_2 \in \partial\Omega$, $\mathbf{n}_f(\mathbf{x}_1) \cdot \mathbf{z} = 1 + \mathbf{n}_f(\mathbf{x}_1) \cdot \mathbf{n}_f(\mathbf{x}_2) > 0$, and if $\mathbf{x}_2 \in \Omega$ than $\mathbf{z} = \mathbf{n}_f(\mathbf{x}_1)$, so in either case $-\mathbf{z}$ is interior in the current configuration. By implicit differentiation,

$$\mathbf{w}'_\alpha(0) = -\,\mathbf{f}(\mathbf{x}_\alpha)^{-1}\mathbf{z} \tag{3.14}$$

showing that $\mathbf{w}'_\alpha(0)$ is the pullback of $-\mathbf{z}$ and is interior in the reference configuration, hence $\mathbf{w}_\alpha(\epsilon) \in \Omega$ for small ϵ. In the case that $\mathbf{x}_2 \in \Omega$ a neighborhood of $\mathbf{x}_2$ is contained in Ω so $\mathbf{w}_2(\epsilon) \in \Omega$ for small ϵ. In both cases the claim holds so we conclude that $S_f \subset \partial\Omega$ and $\mathbf{n}_f(\mathbf{x}_1) + \mathbf{n}_f(\mathbf{x}_2) = \mathbf{0}$ whenever $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$.

If there were more than two points in $\mathbf{f}^{-1}(\mathbf{y})$, then for $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset \mathbf{f}^{-1}(\mathbf{y})$, we have $\mathbf{n}_f(\mathbf{x}_\alpha) + \mathbf{n}_f(\mathbf{x}_\beta) = \mathbf{0}$ for $\alpha, \beta \in \{1, 2, 3\}$ and $\alpha \neq \beta$. Subtracting two of the equations we find that $\mathbf{n}_f(\mathbf{x}_\alpha) - \mathbf{n}_f(\mathbf{x}_\beta) = \mathbf{0}$, implying $\mathbf{n}_f(\mathbf{x}_\alpha) = \mathbf{0}$, which is a contradiction.

That S_f is closed follows from $\mathbf{f} \in C^1(\overline{\Omega}, \mathbb{E}^3)$ and locally invertible. Consider a sequence $\mathbf{x}_1^i \to \mathbf{x}_1$ for $\mathbf{x}_1^i \in S_f$ and $\mathbf{y} = \mathbf{f}(\mathbf{x}_1)$. Then there is a sequence $\mathbf{x}_2^i \in S_f$ such that $\mathbf{f}(\mathbf{x}_1^i) = \mathbf{f}(\mathbf{x}_2^i)$ and $\mathbf{x}_1^i \neq \mathbf{x}_2^i$. Since $\mathbf{f}(\mathbf{x}_2^i) \to \mathbf{y}$ and $\mathbf{f}$ is locally invertible we may conclude that $\mathbf{x}_2^i \to \mathbf{x}_2 \neq \mathbf{x}_1$, thus $\mathbf{x}_1 \in S_f$. □

It is important to identify some properties of the tangent cones, $K_f\mathcal{A}^C_{(in)}$, defined in (3.5). Let

$$K^0_f\mathcal{A}^C_{(in)} = \Big\{\mathbf{h} \in T_f\mathcal{A}^C_{(in)} : \exists\, \delta > 0 \text{ s.t. if } \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2) \text{ and } \mathbf{x}_1 \neq \mathbf{x}_2$$
$$\text{then } \mathbf{h}(\mathbf{x}_1)\cdot\mathbf{n}_f(\mathbf{x}_1) + \mathbf{h}(\mathbf{x}_2)\cdot\mathbf{n}_f(\mathbf{x}_2) + \delta \leq 0\Big\}. \qquad (3.15)$$

Proposition 3.3. *For $\mathbf{f} \in \mathcal{A}^C_{(in)}$, we have $K^0_f\mathcal{A}^C_{(in)} \subset K_f\mathcal{A}^C_{(in)}$ and $K^0_f\mathcal{A}^C_{(in)}$ is relatively open and non-empty. Furthermore, $K_f\mathcal{A}^C_{(in)} \subset \overline{K^0_f\mathcal{A}^C_{(in)}}$.*

Remark 3.1. *The last statement that $K_f\mathcal{A}^C_{(in)} \subset \overline{K^0_f\mathcal{A}^C_{(in)}}$ is not used in Theorem 3.1, and is not true with Lipschitz boundary.*

Proof. Suppose that $\mathbf{h} \in K^0_f\mathcal{A}^C_{(in)}$ with corresponding $\delta > 0$. We want to show that $\mathbf{h} \in K_f\mathcal{A}^C_{(in)}$. For the incompressible case, Corollary 3.1 implies there exists $\mathbf{v} : [0, \epsilon_1) \to \mathcal{A}^M_{in}$, meaning $\mathbf{v}_\epsilon \in W^{2,p}(\Omega, \mathbb{E}^3)$ satisfies incompressibility and the boundary conditions for $\epsilon \in [0, \epsilon_1)$, $\mathbf{v}_0 = \mathbf{f}$, and $\frac{d}{d\epsilon}\mathbf{v}_0 = \mathbf{h}$. In the compressible case, $\mathbf{v}_\epsilon = \mathbf{f} + \epsilon\mathbf{h}$ is locally injective if ϵ is sufficiently small. This follows from continuity of the determinant operator, H, mapping $W^{2,p}(\Omega, \mathbb{E}^3) \to C(\overline{\Omega}, \mathbb{E}^3)$ and the fact that $\det(\nabla\mathbf{f})$ is uniformly bounded from below. We must show that $\mathbf{v}_\epsilon$ is globally injective on Ω for small ϵ.

Suppose for contradiction that there is a sequence $\epsilon^i \to^+ 0$ and $\{\mathbf{x}^i_1, \mathbf{x}^i_2\} \subset \Omega$ such that $\mathbf{v}_{\epsilon^i}(\mathbf{x}^i_1) = \mathbf{v}_{\epsilon^i}(\mathbf{x}^i_2)$ and $\mathbf{x}^i_1 \neq \mathbf{x}^i_2$. Then we restrict to a subsequence (without re-indexing) such that $\mathbf{x}^i_\alpha \to \mathbf{x}_\alpha$ for $\alpha \in \{1, 2\}$ and $\mathbf{x}^i_\alpha \in \overline{\Omega}$. By continuous dependence of $\mathbf{v}$ on $\mathbf{x}$ and ϵ, $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$.

We now consider two cases, first suppose that $\mathbf{x}_1 = \mathbf{x}_2$. There is some $0 < \epsilon_2 < \epsilon_1$ such that $|\nabla\mathbf{v}_\epsilon(\mathbf{x})\mathbf{w}| \geq \delta'|\mathbf{w}|$ for some $\delta' > 0$. Since $\overline{\Omega} \times [0, \epsilon_2]$ is compact, $\nabla\mathbf{v}_\epsilon(\mathbf{x})$ is uniformly continuous with respect to $\mathbf{x}$ and ϵ, and in particular the remainder of the Taylor expansion is independent of ϵ. For $\{\mathbf{x}_1^i, \mathbf{x}_2^i\} \subset \Omega$,

$$\left|\mathbf{v}_\epsilon(\mathbf{x}_1^i) - \mathbf{v}_\epsilon(\mathbf{x}_2^i)\right| \geq \left|\nabla\mathbf{v}_\epsilon(\mathbf{x}_1^i)[\mathbf{x}_2^i - \mathbf{x}_1^i]\right| - o(|\mathbf{x}_2^i - \mathbf{x}_1^i|). \tag{3.16}$$

We choose r such that the error satisfies $o(r) < \delta' r$, which implies that $\mathbf{v}_\epsilon$ is injective on balls of radius r for $\epsilon \in [0, \epsilon_2]$. Then we choose i large enough that $|\mathbf{x}_\alpha^i - \mathbf{x}_\alpha| < \frac{r}{2}$ and $\epsilon^i < \epsilon_2$. Since $\mathbf{x}_1^i$ and $\mathbf{x}_2^i$ are contained in a ball of radius r, this contradicts that $\mathbf{v}_{\epsilon^i}(\mathbf{x}_1^i) = \mathbf{v}_{\epsilon^i}(\mathbf{x}_2^i)$.

For the second case, suppose $\mathbf{x}_1 \neq \mathbf{x}_2$. Lemma 3.1 implies that $\{\mathbf{x}_1, \mathbf{x}_2\} \subset \partial\Omega$ and $\mathbf{n}_f(\mathbf{x}_1) + \mathbf{n}_f(\mathbf{x}_2) = \mathbf{0}$. We extend $\mathbf{v}_\epsilon$ to $\tilde{\Omega}$, a neighborhood of $\overline{\Omega}$, so that $\nabla\mathbf{v}_\epsilon$ is uniformly continuous in $\mathbf{x}$ and ϵ. For large enough i, $\mathbf{v}_\epsilon$ is now defined on the line connecting $\mathbf{x}_\alpha^i$ and $\mathbf{x}_\alpha$.

$$\begin{aligned}\sum_{\alpha=1}^{2} \mathbf{n}_f(\mathbf{x}_\alpha) \cdot \mathbf{v}_\epsilon(\mathbf{x}_\alpha^i) &= \sum_{\alpha=1}^{2} \int_0^1 \mathbf{n}_f(\mathbf{x}_\alpha) \cdot \nabla\mathbf{v}_\epsilon(t\mathbf{x}_\alpha^i + (1-t)\mathbf{x}_\alpha)[\mathbf{x}_\alpha^i - \mathbf{x}_\alpha]dt \\ &\quad + \sum_{\alpha=1}^{2} \mathbf{n}_f(\mathbf{x}_\alpha) \cdot \mathbf{v}_\epsilon(\mathbf{x}_\alpha) \\ &\leq -\epsilon\delta + \|\nabla\mathbf{v}_\epsilon\|_{C(\tilde{\Omega}, L(\mathbb{E}^3))} \sum_{\alpha=1}^{2} |\mathbf{x}_\alpha^i + \mathbf{x}_\alpha| - o(\epsilon). \end{aligned} \tag{3.17}$$

We choose i large enough that $o(\epsilon^i) < \epsilon^i \frac{\delta}{4}$ and $|\mathbf{x}_\alpha^i - \mathbf{x}_\alpha| < \frac{\delta}{8}\|\nabla\mathbf{v}_\epsilon\|^{-1}_{C(\tilde{\Omega}, L(\mathbb{E}^3))}$ which makes (3.17) negative. This contradicts that $\mathbf{v}_{\epsilon^i}(\mathbf{x}_1^i) = \mathbf{v}_{\epsilon^i}(\mathbf{x}_2^i)$, in which case the left side of (3.17) is 0.

It is clear that $K_f^0\mathcal{A}_{(in)}^C$ is relatively open by uniform continuity of $\mathbf{h} \in W^{2,p}(\Omega, \mathbb{E}^3)$. We construct the interior displacement from a smooth interior vector field $\mathbf{j}$, from

Lemma C.5. There exists a neighborhood of Γ, U, such that if $\mathbf{x}_1 \in S_f \cap U$ and $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$ for $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{x}_2 \notin \Gamma$ by injectivity of $\mathbf{f}_0$ on $\overline{\Omega}$. Then $S_f \backslash (S_f \cap U)$ is compact and it follows that $\mathbf{h}_1(\mathbf{x}_1) \cdot \mathbf{n}_f(\mathbf{x}_1) + \mathbf{h}_2(\mathbf{x}_2) \cdot \mathbf{n}_f(\mathbf{x}_2) + \delta \leq 0$ some $\delta > 0$.

With the incompressibility constraint, $\mathbf{j}$ must be adjusted to a field $\mathbf{h} \in W^{2,p}(\Omega, \mathbb{E}^3)$, with compatible boundary conditions and satisfying $\operatorname{cof}(\nabla \mathbf{f}) \cdot \nabla \mathbf{h} = 0$. By Lemma 2.6 there exists $\mathbf{u}$ with Dirichlet boundary conditions solving $\operatorname{cof}(\nabla \mathbf{f}) \cdot \nabla \mathbf{u} = \operatorname{cof}(\nabla \mathbf{f}) \cdot \nabla \mathbf{j}$ if

$$0 = \int_\Omega \operatorname{cof}(\nabla \mathbf{f}) \cdot \nabla \mathbf{j} \, dV = \int_{\partial\Omega} \operatorname{cof}(\nabla \mathbf{f})^\top \mathbf{j} \cdot \boldsymbol{\nu} \, dS. \tag{3.18}$$

Then setting $\mathbf{h} = \mathbf{j} - \mathbf{u}$ satisfies $\mathbf{h} \in K_f^0 \mathcal{A}_{in}^C$. Condition (3.18), however, is always failed by an interior vector field. The set $\partial\Omega \backslash (S_f \cup \Gamma)$ is relativily open and non-empty so we adjust $\mathbf{j}$ in a neighborhood of a point in that set to satisfy the compatibility condition. To find $\mathbf{x} \in \partial\Omega \backslash (S_f \cup \Gamma)$, we maximize the distance of $\mathbf{f}(\mathbf{x})$ from $\mathbf{f}_0(\Gamma)$. The maximum is attained at $\mathbf{x} \in \partial\Omega \backslash \Gamma$ and $\mathbf{x} \notin S_f$ else there would be a point in Ω closer to Γ.

To show $K_f \mathcal{A}_{(in)}^C \subset \overline{K_f^0 \mathcal{A}_{(in)}^C}$, suppose that $\mathbf{h} \in K_f \mathcal{A}_{(in)}^C$. We first claim that $\mathbf{h}$ satisfies $\mathbf{h}_1(\mathbf{x}_1) \cdot \mathbf{n}_f(\mathbf{x}_1) + \mathbf{h}_2(\mathbf{x}_2) \cdot \mathbf{n}_f(\mathbf{x}_2) \leq 0$ whenever $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$. Then we consider $\mathbf{h}_k = \mathbf{h} + k^{-1}\mathbf{j}$, where $\mathbf{j} \in K_f^0 \mathcal{A}_{(in)}^C$. It is easy to see that $\mathbf{h}_k \in K_f^0 \mathcal{A}_{(in)}^C$ and $\mathbf{h}_k \to \mathbf{h}$ as $k \to \infty$. To prove the first claim we contradict injectivity of $\mathbf{v}_\epsilon$ with an argument similar to the one in Lemma 3.1. Suppose that $\mathbf{h}_1(\mathbf{x}_1) \cdot \mathbf{n}_f(\mathbf{x}_1) + \mathbf{h}_2(\mathbf{x}_2) \cdot \mathbf{n}_f(\mathbf{x}_2) > 0$ and $\mathbf{y} = \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$. We find two interior solutions, $\mathbf{x} = \mathbf{w}_\alpha(\epsilon)$ for $\alpha \in \{1, 2\}$, to

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \epsilon) = \mathbf{v}_\epsilon(\mathbf{x}) - \mathbf{y} - \epsilon \frac{1}{2}(\mathbf{h}(\mathbf{x}_1) + \mathbf{h}(\mathbf{x}_2)).$$

The path of solutions follows from the implicit function theorem, and

$$\mathbf{w}'_\alpha(0) = -\nabla\mathbf{f}(\mathbf{x}_\alpha)^{-1}\Big(- \frac{1}{2}\mathbf{h}(\mathbf{x}_1) - \frac{1}{2}\mathbf{h}(\mathbf{x}_2) + \mathbf{h}(\mathbf{x}_\alpha)\Big).$$

Then

$$\begin{aligned}
&\Big(- \frac{1}{2}\mathbf{h}(\mathbf{x}_1) - \frac{1}{2}\mathbf{h}(\mathbf{x}_2) + \mathbf{h}(\mathbf{x}_\alpha)\Big) \cdot \mathbf{n}_f(\mathbf{x}_\alpha)\\
=&\frac{1}{2}\Big(\mathbf{h}(\mathbf{x}_1)\cdot\mathbf{n}_f(\mathbf{x}_1) + \mathbf{h}(\mathbf{x}_2)\cdot\mathbf{n}_f(\mathbf{x}_2)\Big)\\
&+\Big(\frac{1}{2}\mathbf{h}(\mathbf{x}_1) + \frac{1}{2}\mathbf{h}(\mathbf{x}_2) - \frac{1}{2}\mathbf{h}(\mathbf{x}_\alpha)\Big)\cdot(\mathbf{n}_f(\mathbf{x}_1) + \mathbf{n}_f(\mathbf{x}_2))\\
>&0,
\end{aligned}$$

and it follows that $\mathbf{w}_\alpha(\epsilon) \in \Omega$ for small $\epsilon > 0$, which contradicts injectivity of $\mathbf{v}_\epsilon$ on Ω. □

Theorem 3.1. *Given that $\partial\Omega$ is C^1 and the assumptions from Section 3.2, there exists a minimizer $\mathbf{f}^* \in \mathcal{A}^C_{(in)}$. For any minimizer, there exists $\psi \in \mathcal{M}(\partial\Omega) = C(\partial\Omega)^*$, with $\psi \geq 0$ and $\langle\psi,\phi\rangle_{C(\partial\Omega)} = 0$ if $\phi(\mathbf{x}_1) + \phi(\mathbf{x}_2) = 0$ whenever $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$. (This implies that* $\operatorname{supp}\psi \subset S_f$ *and in a sense $\psi(\mathbf{x}_1) = \psi(\mathbf{x}_2)$ if $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$.) Moreover, the weak equilibrium equation is satisfied:*

$$\langle DE[\mathbf{f}^*],\mathbf{h}\rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} + \langle\psi,\mathbf{h}\cdot\mathbf{n}_{f^*}\rangle_{C(\partial\Omega)} = 0 \tag{3.19}$$

for all $\mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)}$. With incompressibility, there is also the Lagrange multiplier $p \in W^{1,p}(\Omega)^*$ such that

$$\langle DE[\mathbf{f}^*],\mathbf{h}\rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} + \langle p, DH[\mathbf{f}^*]\mathbf{h}\rangle_{W^{1,p}(\Omega)} + \langle\psi,\mathbf{h}\cdot\mathbf{n}_{f^*}\rangle_{C(\partial\Omega)} = 0 \tag{3.20}$$

for all $\mathbf{h} \in T_{f^*}\mathcal{A}^C$.

Proof. Suppose that $\mathbf{f}^*$ is an energy minimizer, cf. Proposition 3.1 . We now show that equation (3.19) is satisfied. Consider $M_{f^*} \subset \mathbb{R} \times C(\partial\Omega)$ defined by

$$M_{f^*} = \Big\{(l, z) : \exists\, \mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)} \ s.t.\ \langle DE[\mathbf{f}^*], \mathbf{h}\rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} \leq l,$$
$$\text{and if } \mathbf{f}^*(\mathbf{x}) = \mathbf{f}^*(\mathbf{y}) \text{ then } \mathbf{h}(\mathbf{x}) \cdot \mathbf{n}_{f^*}(\mathbf{x}) + \mathbf{h}(\mathbf{y}) \cdot \mathbf{n}_{f^*}(\mathbf{y}) \leq z(\mathbf{x}) + z(\mathbf{y})\Big\}. \tag{3.21}$$

The origin is contained in M_{f^*} by selecting $\mathbf{h} = \mathbf{0}$. Given $(l_\alpha, z_\alpha) \in M_{f^*}$ for $\alpha \in \{1, 2\}$, we consider a convex combination $(l, z) = (1 - \gamma)(l_1, z_1) + \gamma(l_2, z_2)$ with $\gamma \in [0, 1]$. Then the convex combination of displacements $(1 - \gamma)\mathbf{h}_1 + \gamma\mathbf{h}_2 \in T_f\mathcal{A}^C_{(in)}$, and linearity of the criterion of (3.21) shows that $(l, z) \in M_{f^*}$ and M_{f^*} is convex.

Suppose for some $\delta > 0$, $l \leq -\delta$ and $z(\mathbf{x}) \leq -\delta$ for all $\mathbf{x} \in \partial\Omega$, then $(l, z) \notin M_{f^*}$. Otherwise, by Proposition 3.3 the corresponding displacement satisfies $\mathbf{h} \in K_{f^*}\mathcal{A}^C_{(in)}$ (i.e. there exists an admissible variation in the $\mathbf{h}$ direction) and Proposition 3.2 contradicts that $l < 0$ (due to local optimality of $\mathbf{f}^*$).

Let

$$M^- = \Big\{(l, z) \in \mathbb{R} \times C(\partial\Omega) : l \leq 0 \text{ and } z(\mathbf{x}) \leq 0\ \forall\, \mathbf{x} \in \partial\Omega\Big\}.$$

We have shown M_{f^*} contains no interior points of M^-. The separating hyperplane theorem implies (see Theorem 3 (Eidelheit Separation Theorem), Section 5.12 in [30]) the existence of $(\lambda_0, \psi_0) \in \mathbb{R} \times C(\partial\Omega)^*$, not identically zero, separating M_{f^*} from M^- in the sense that

$$\lambda_0 l + \langle\psi_0, z\rangle_{C(\partial\Omega)} \geq 0\ \forall\, (l, z) \in M_{f^*} \tag{3.22}$$

$$\lambda_0 l + \langle\psi_0, z\rangle_{C(\partial\Omega)} \leq 0\ \forall\, (l, z) \in M^-. \tag{3.23}$$

Equation (3.23) implies $\lambda_0 \geq 0$ and $\langle \psi_0, \phi \rangle_{C(\partial\Omega)} \geq 0$ whenever $\phi(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \partial\Omega$.

Then we claim that $\lambda_0 > 0$. This follows from the existence of $(l, z) \in M_{f^*}$ with $z < 0$. For that we need to find $\mathbf{h} \in K^0_{f^*}\mathcal{A}^C_{(in)}$, which satisfies $\mathbf{h}(\mathbf{x}_1) \cdot \mathbf{n}_{f^*}(\mathbf{x}_1) + \mathbf{h}(\mathbf{x}_2) \cdot \mathbf{n}_{f^*}(\mathbf{x}_2) + \delta \leq 0$ whenever $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$, and that is done in Proposition 3.3. We select $z(\mathbf{x}) = \frac{-\delta}{2}$ and $(l, z) \in M_{f^*}$ with $l = \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega, \mathbb{E}^3)}$. If $\lambda_0 = 0$ then $\lambda_0 l + \langle \psi_0, z \rangle_{C(\partial\Omega)} < 0$ since $\psi_0 \neq 0$ and $\psi_0 \geq 0$, but this contradicts (3.22). We now set $\psi = \psi_0/\lambda_0$.

Suppose that $z \in C(\partial\Omega)$ satisfies $z(\mathbf{x}_1) + z(\mathbf{x}_2) = 0$ whenever $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$. Then choosing $\mathbf{h} \equiv \mathbf{0}$, it is immediate from the definition of M_{f^*} that

$$\langle \psi, z \rangle_{C(\partial\Omega)} = 0. \tag{3.24}$$

Given $\mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)}$, let $l = \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega, \mathbb{E}^3)}$ and $z(\mathbf{x}) = \mathbf{h}(\mathbf{x}) \cdot \mathbf{n}_{f^*}(\mathbf{x})$. Then $(l, z) \in M_{f^*}$ and

$$\begin{aligned} 0 \leq& l + \langle \psi, z \rangle_{C(\partial\Omega)} \\ \leq& \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega, \mathbb{E}^3)} + \langle \psi, \mathbf{n} \cdot \mathbf{h} \rangle_{C(\partial\Omega)} \end{aligned} \tag{3.25}$$

and the opposite inequality follows from the same argument with $-\mathbf{h}$.

Finally, for incompressibility, a final step with the closed range theorem implies the existence of a pressure satisfying (3.20). □

Next we generalize to the case that $\partial\Omega$ is Lipschitz, or equivalently $\partial\Omega$ satisfies the cone condition (see Lemma C.3). We use the same definition for S_f, (3.11), and repeat Lemma 3.1 and Proposition 3.3. In place of the normal vectors we refer to the interior cone (the same as (3.5) in the finite-dimensional setting) defined by

$$K_x\overline{\Omega} = \left\{\mathbf{w}_0 \in \mathbb{E}^3 : \exists\ \epsilon_1 > 0 \text{ and } \mathbf{w} \in C([0,\epsilon_1), \overline{\Omega}) \text{ s.t. } \mathbf{w}(0) = \mathbf{x} \text{ and } \mathbf{w}'(0) = \mathbf{w}_0\right\}. \quad (3.26)$$

We also push the cones forward into the current configuration by

$$\mathbf{f}_* K_x\overline{\Omega} = \left\{\nabla\mathbf{f}(\mathbf{x})\mathbf{w} : \mathbf{w} \in K_x\overline{\Omega}\right\}. \quad (3.27)$$

Lemma 3.2. *Let $\mathbf{f} \in \mathcal{A}^C$. For any $\mathbf{y} \in \mathbf{f}(S_f)$, the cardinality of $\mathbf{f}^{-1}(\mathbf{y})$ is finite and bounded by a constant depending on $E[\mathbf{f}]$ as well as Ω. If $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$ for $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{f}_* K_{x_1}\overline{\Omega} \cap \mathbf{f}_* K_{x_2}\overline{\Omega}$ has empty interior. Again S_f is closed.*

Proof. Given $\mathbf{x}_1 \in S_f$ let $\mathbf{x}_2 \in \overline{\Omega}$ be such that $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2) = \mathbf{y}$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. We claim that since $\mathbf{f}$ is injective on Ω, $\mathbf{f}_* K_{x_1}\overline{\Omega} \cap \mathbf{f}_* K_{x_2}\overline{\Omega}$ does not contain an interior point. We extend $\mathbf{f}$ outside of $\overline{\Omega}$ and suppose that $\mathbf{z} \in \operatorname{int} \mathbf{f}_* K_{x_1}\overline{\Omega} \cap \mathbf{f}_* K_{x_2}\overline{\Omega}$. Then, as in Lemma 3.1, we solve $\mathbf{f}(\mathbf{w}) = \mathbf{y} + \epsilon\mathbf{z}$. The same calculation (see (3.14)) implies the solutions satisfy $\mathbf{w}'_\alpha(0) \in \operatorname{int} K_{x_\alpha}\overline{\Omega}$ for $\alpha \in \{1,2\}$, hence $\mathbf{w}_\alpha(\epsilon) \in \Omega$ for $\epsilon > 0$ contradicting injectivity of $\mathbf{f}$.

Next we consider the volumes given by $m(K) = \mathrm{Vol}(\{\mathbf{y} \in K : |\mathbf{y}| \leq 1\})$ and claim there are cones $C_x \subset \operatorname{int} \mathbf{f}_* K_x\overline{\Omega}$ such that $m(C_x) \geq \gamma(\Omega, E[\mathbf{f}])$. This follows from the supremum bound on $\nabla\mathbf{f}^{-1}$, (3.8), and the uniform cone condition on Ω. There is some n such that $n\gamma \geq \frac{4}{3}\pi r^3$ and we can conclude that the cardinality of $\mathbf{f}^{-1}(\mathbf{y})$ is less than or equal to n. Else, since the interior of the pushforward cones are disjoint, $m(\cup_{\mathbf{x}\in\mathbf{f}^{-1}(\mathbf{y})}\mathbf{f}_* K_x\overline{\Omega}) \geq \sum_{\mathbf{x}\in\mathbf{f}^{-1}(\mathbf{y})} m(C_x) > \frac{4}{3}\pi r^3$ which contradicts the volume bound from the unit ball $m(\cup_{\mathbf{x}\in\mathbf{f}^{-1}(\mathbf{y})}\mathbf{f}_* K_x\overline{\Omega}) \leq \mathrm{Vol}(B_1)$.

Suppose $\mathbf{x}_1^i \to \mathbf{x}$ for $\mathbf{x}_1^i \in S_f$. Then choose $\mathbf{x}_2^i \neq \mathbf{x}_1^i$ and $\mathbf{f}(\mathbf{x}_1^i) = \mathbf{f}(\mathbf{x}_2^i)$. By compactness of $\partial\Omega$ there is a subsequence with $\mathbf{x}_2^i \to \mathbf{x}_2$ and by continuity of $\mathbf{f}$, $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$. It remains to show that $\mathbf{x}_1 \neq \mathbf{x}_2$ but again this follows from local invertibility of $\mathbf{f}$. □

In the following proposition we characterize a convex subset of the interior cone $K_f\mathcal{A}^C_{(in)}$ in terms of the local information of $\mathbf{f}$ and Ω. This subset is analogous to the interior of the convex subsets of the finite dimensional cones given by

$$\hat{K}_x\overline{\Omega} = \left\{\mathbf{w} \in \mathbb{E}^3 : \exists\, \mathbf{z} \in C(\partial\Omega, \mathbb{E}^3) \text{ s.t. } \mathbf{z}(\mathbf{x}) = \mathbf{w}, \text{ and } \mathbf{z}(\mathbf{y}) \in K_y\overline{\Omega}\ \forall\ \mathbf{y} \in \partial\Omega\right\}. \tag{3.28}$$

We define the pushforward, $\mathbf{f}_*\hat{K}_x\overline{\Omega}$, in the same manner as (3.27).

Proposition 3.4. *The tangent cone $K_f\mathcal{A}^C_{(in)}$ contains*

$$\begin{aligned} K_f^0\mathcal{A}^C_{(in)} = \Big\{\mathbf{h} \in T_f\mathcal{A}^C_{(in)} : \exists\, \delta > 0 \text{ s.t. if } \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2) \text{ for } \mathbf{x}_1 \neq \mathbf{x}_2,\\ \text{then } \mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2) + \mathbf{w} \in \mathbf{f}_*\hat{K}_{x_1}\overline{\Omega} - \mathbf{f}_*\hat{K}_{x_2}\overline{\Omega} \text{ for all } \mathbf{w} \in \mathbb{E}^3 \text{ s.t. } |\mathbf{w}| < \delta\}, \end{aligned} \tag{3.29}$$

and $K_f^0\mathcal{A}^C_{(in)}$ is non-empty.

Proof. Suppose that $\mathbf{h} \in K_f^0\mathcal{A}^C_{(in)}$. The proof is nearly identical to Proposition 3.3 using Lemma 3.2 in place of Lemma 3.1 and adjusting the argument to use interior cones rather than normal vectors.

As in Proposition 3.3, to show that $\mathbf{h} \in K_f\mathcal{A}^C_{(in)}$, we suppose $\epsilon^i \to^+ 0$, $\mathbf{v}_{\epsilon^i}(\mathbf{x}_1^i) = \mathbf{v}_{\epsilon^i}(\mathbf{x}_2^i)$ with $\mathbf{x}_1^i \neq \mathbf{x}_2^i$, and $\mathbf{x}_\alpha^i \to \mathbf{x}_\alpha$. We consider the case that $\mathbf{x}_1 \neq \mathbf{x}_2$. Then $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$ and $\{\mathbf{x}_1, \mathbf{x}_2\} \subset S_f$ so we have that $\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2) \in \operatorname{int} \mathbf{f}_*\hat{K}_{x_1}\overline{\Omega} - \operatorname{int} \mathbf{f}_*\hat{K}_{x_2}\overline{\Omega}$. Thus there are non-empty open cones $C_1 \subset \text{int } \mathbf{f}_* \hat{K}_{x_1}\overline{\Omega}$ and $C_2 \subset \text{int } \mathbf{f}_* \hat{K}_{x_2}\overline{\Omega}$ satisfying $\mathbf{x}_\alpha + C_\alpha^{\delta'} \subset \Omega$ for $C_\alpha^{\delta'} = \{\mathbf{w} \in C_\alpha : |\mathbf{w}| \leq \delta'\}$ and $\delta' > 0$. Furthermore, there is $\delta'' > 0$ such that $\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2) + \mathbf{w} \in C_1 - C_2$ for all $|\mathbf{w}| \leq \delta''$. Since C_1 and C_2 are convex and disjoint they are separated by a plane. Let $\mathbf{n}$ be a unit vector such that $\mathbf{n} \cdot \mathbf{w} \geq 0$ whenever $\mathbf{w} \in C_1 - C_2$. Then we extend $\mathbf{v}_\epsilon$ to a neighborhood of $\overline{\Omega}$ containing the line connecting $\mathbf{x}_\alpha^i$ and $\mathbf{x}_\alpha$ for large i and $\alpha \in \{1, 2\}$, and

$$\begin{aligned}\mathbf{n} \cdot (\mathbf{v}_\epsilon(\mathbf{x}_1^i) - \mathbf{v}_\epsilon(\mathbf{x}_2^i)) = & \int_0^1 \mathbf{n} \cdot \Big(\nabla \mathbf{v}_\epsilon(t\mathbf{x}_1^i + (1-t)\mathbf{x}_1)(\mathbf{x}_1^i - \mathbf{x}_1)\Big) dt \\ & - \int_0^1 \mathbf{n} \cdot \Big(\mathbf{v}_\epsilon(t\mathbf{x}_2^i + (1-t)\mathbf{x}_2)(\mathbf{x}_2^i - \mathbf{x}_2)\Big) dt \\ & + \mathbf{n} \cdot (\mathbf{v}_\epsilon(\mathbf{x}_1) - \mathbf{v}_\epsilon(\mathbf{x}_2)) \\ \geq & \epsilon\delta - \|\nabla \mathbf{v}_\epsilon\|_{C(\tilde{\Omega}, L(\mathbb{E}^3))} \sum_{\alpha=1}^{2} |\mathbf{x}_\alpha^i - \mathbf{x}_\alpha| - o(\epsilon).\end{aligned}$$

In the last step we chose $\mathbf{w} = -\delta''\mathbf{n}$ and used $\mathbf{n} \cdot (\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2) + \mathbf{v}) \geq 0$. This contradicts that $\mathbf{v}_{\epsilon^i}(\mathbf{x}_1^i) = \mathbf{v}_{\epsilon^i}(\mathbf{x}_2^i)$ for large i.

The proof in Proposition 3.3 that $K_f^0 \mathcal{A}_{(in)}^C$ is non-empty did not use additional regularity of the boundary. □

Now we restate Theorem 3.1 and give a proof assuming that $\partial\Omega$ is Lipschitz but not necessarily C^1.

Theorem 3.2. *Given the assumptions of section 3.2 with $\partial\Omega$ Lipschitz, there exists a minimizer $\mathbf{f}^* \in \mathcal{A}_{(in)}^C$. For any minimizer, there exists $\boldsymbol{\psi} \in \mathcal{M}(\partial\Omega, \mathbb{E}^3) = C(\partial\Omega, \mathbb{E}^3)^*$ satisfying the following:*

- $\langle \boldsymbol{\psi}, \mathbf{z} \rangle \leq 0$ *for* $\mathbf{z} \in C(\partial\Omega, \mathbb{E}^3)$ *such that* $\mathbf{z}(\mathbf{x}) \in \mathbf{f}_*^* K_x(\overline{\Omega})$ *for all* $\mathbf{x} \in \partial\Omega$,

- $\langle \boldsymbol{\psi}, \mathbf{z} \rangle = 0$ *if* $\mathbf{z} \in C(\partial\Omega, \mathbb{E}^3)$ *satisfies* $\mathbf{z}(\mathbf{x}_1) = \mathbf{z}(\mathbf{x}_2)$ *whenever* $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$*,*
- *the equilibrium equation is satisfied:*

$$\langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} + \langle \boldsymbol{\psi}, \mathbf{h} \rangle_{C(\partial\Omega,\mathbb{E}^3)} = 0 \tag{3.30}$$

for all $\mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)}$*. With incompressibility, there is also the Lagrange multiplier* $p \in W^{1,p}(\Omega)^*$ *such that for all* $\mathbf{h} \in T_{f^*}\mathcal{A}^C$*,*

$$\langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} + \langle p, DH[\mathbf{f}^*]\mathbf{h} \rangle_{W^{1,p}(\Omega)} + \langle \boldsymbol{\psi}, \mathbf{h} \rangle_{C(\partial\Omega,\mathbb{E}^3)} = 0. \tag{3.31}$$

Proof. Suppose that $\mathbf{f}^*$ is a minimizer, Proposition 3.1 still applies for existence, and let us show that equation (3.30) is satisfied. We consider $M^+_{f^*}, M^-_{f^*} \subset \mathbb{R} \times C(\partial\Omega, \mathbb{E}^3)$ defined by

$$\begin{aligned} M^+_{f^*} = \Big\{ (l, \mathbf{z}) : \exists\, \mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)}\ s.t.\ &\langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} \le l \\ &\text{and if } \mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2) \text{ then} \\ &\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2) - \mathbf{z}(\mathbf{x}_1) + \mathbf{z}(\mathbf{x}_2) \in \mathbf{f}^*_* \hat{K}_{x_1}\overline{\Omega} - \mathbf{f}^*_* \hat{K}_{x_2}\overline{\Omega} \Big\}, \end{aligned} \tag{3.32}$$

$$M^-_{f^*} = \Big\{ (l, \mathbf{z}) : l \le 0,\ \mathbf{z}(\mathbf{x}) \in \mathbf{f}^*_* K_x \overline{\Omega}\ \forall\, \mathbf{x} \in \partial\Omega \Big\}. \tag{3.33}$$

Just as in Theorem 3.19, $M^+_{f^*}$ and $M^-_{f^*}$ are convex cones containing the origin. Convexity of $M^-_{f^*}$ follows from Lemma C.1. $M^-_{f^*}$ has non-empty interior by Lemma C.5.

Suppose $(l, \mathbf{z}) \in \operatorname{int} M^-_{f^*}$, then $(l, \mathbf{z}) \notin M^+_{f^*}$. This follows from Proposition 3.4 as $(l, \mathbf{z}) \in \operatorname{int} M^-_{f^*}$ implies that $\mathbf{h} \in K^0_{f^*}\mathcal{A}^C_{(in)}$ and Proposition 3.2 contradicts that $l < 0$.

Since the interior of $M^{-}_{f^*}$ is non-empty and disjoint from $M^{+}_{f^*}$, the separating hyperplane theorem implies the existence of $(\lambda_0, \boldsymbol{\psi}_0) \in \mathbb{R} \times C(\partial\Omega, \mathbb{E}^3)^*$ separating $M^{+}_{f^*}$ from $M^{-}_{f^*}$ in the sense that $\lambda_0 l + \langle \boldsymbol{\psi}_0, \mathbf{z} \rangle_{C(\partial\Omega,\mathbb{E}^3)} \geq 0$ for all $(l, \mathbf{z}) \in M^{+}_{f^*}$, and the opposite inequality holds for $(l, \mathbf{z}) \in M^{-}_{f^*}$.

Next we claim that $\lambda_0 > 0$. This follows from the existence of an interior point $(l, \mathbf{z}) \in M^{+}_{f^*}$ with $\mathbf{z} \in \operatorname{int} M^{-}_{f^*}$, i.e. $\mathbf{z}(\mathbf{x}) \in \operatorname{int} \hat{K}_x\overline{\Omega}$ for all $\mathbf{x}$. We have shown there exists $\mathbf{h} \in K^0_{f^*}\mathcal{A}^C_{(in)}$ in Proposition 3.4. Let $\mathbf{z} \in C(\partial\Omega, \mathbb{E}^3)$ be everywhere interior, cf. Lemma C.5, and suppose $|\mathbf{z}(\mathbf{x})| \leq \delta/2$ for all $\mathbf{x} \in \partial\Omega$ with δ corresponding to $\mathbf{h}$ as in the definition of $K^0_{f^*}\mathcal{A}^C_{(in)}$. Then it follows that $\mathbf{h}(\mathbf{x}_1)-\mathbf{h}(\mathbf{x}_2)-\mathbf{z}(\mathbf{x}_1)+\mathbf{z}(\mathbf{x}_2) \in \mathbf{f}^*_*\hat{K}_{x_1}\overline{\Omega}-\mathbf{f}^*_*\hat{K}_{x_2}\overline{\Omega}$ whenever $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$, so for $l \geq \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)}$, $(l, \mathbf{z}) \in M^{+}_{f^*}$. This implies that $\lambda_0 > 0$ and we let $\boldsymbol{\psi} = \boldsymbol{\psi}_0/\lambda_0$

Suppose that $\mathbf{z} \in C(\partial\Omega, \mathbb{E}^3)$ satisfies $\mathbf{z}(\mathbf{x}_1) = \mathbf{z}(\mathbf{x}_2)$ whenever $\mathbf{f}^*(\mathbf{x}_1) = \mathbf{f}^*(\mathbf{x}_2)$, then using $\mathbf{h} = \mathbf{0}$ we find that $(0, \mathbf{z}) \in M^{+}_{f^*}$, hence $\langle \boldsymbol{\psi}, \mathbf{z} \rangle_{C(\partial\Omega,\mathbb{E}^3)} = 0$.

Given $\mathbf{h} \in T_{f^*}\mathcal{A}^C_{(in)}$, we choose $l = \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)}$ and $\mathbf{z}(\mathbf{x}) = \mathbf{h}(\mathbf{x})$, which implies that $(l, \mathbf{z}) \in M^{+}_{f^*}$ and

$$0 \leq \langle DE[\mathbf{f}^*], \mathbf{h} \rangle_{W^{2,p}(\Omega,\mathbb{E}^3)} + \langle \boldsymbol{\psi}, \mathbf{h} \rangle_{C(\partial\Omega,\mathbb{E}^3)}. \tag{3.34}$$

The opposite inequality follows from the same argument with $-\mathbf{h}$.

Finally, with incompressibility a final step with the closed range theorem implies the existence of a pressure. □

3.5 Conclusion

We develop a novel approach to the problem of self-contact in non-linear elasticity. We relate the displacements to the space of continuous functions on the boundary and the topology of uniform continuity, which is a natural choice for this problem because of the well behaved positive cones. Although we do not pose the constraint via a globally defined function on the space of admissible deformations, the interior cones defined in (3.15) and (3.29) provide all the necessary local information about the constraint. We take full advantage of the additional freedom allowed by the inequality type constraint. Indeed, were we required to show that a linearization of an operating posing the constraint is surjective, as we did in Chapter 2, we would face serious problems. We might try to work in the function space that is the trace of $W^{2,p}(\Omega, \mathbb{E}^3)$ functions, but would find that the normal vector in the current configuration does not posses sufficient differentiability. Instead, working with continuous functions not only simplifies the analysis, but also strengthens the result as we obtain a Lagrange multiplier in a measure space as opposed to some larger dual space.

The work in this chapter furthers our goal of developing a consistent well-posed theory for second-gradient elasticity, but also has broader implications. For example in classical elasticity with a global injectivity constraint, our results imply that if an energy minimizer satisfies, $\mathbf{f}^* \in C^1(\overline{\Omega}, \mathbb{E}^3)$ and $\det(\nabla \mathbf{f}^*(\mathbf{x})) \geq \delta > 0$ for all $\mathbf{x} \in \overline{\Omega}$, then there exists a measure-valued surface traction. It would not be difficult to show that if an equilibrium solution possesses enough regularity to be a strong solution, then the surface traction would have similar higher regularity. We do not expect so much regularity of equilibrium solutions with non-trivial coincidence sets as full regularity does not hold for the simplest scalar variational inequalities. It is interesting to note that even when the equilibrium is smooth along with the stored energy function and the boundary of the domain, if the coincidence set is non-trivial then solutions to the linearized equations may possess singularities. This observation shows the difficulty of carrying out linearized analysis and suggests that other types of non-linear analysis are needed to progress with this problem.

An ulterior motivation for studying the self-contact problem is to understand an infinite-dimensional application of degree theory for non-linear complementarity problems. There are degrees well suited for obstacle problems, see for example [27]. The work of Le is restricted to constraints by convex subsets of Banach spaces, as is often the case with the study of obstacle problems. We believe that our approach provides an interesting context to study non-convex constraints.

CHAPTER 4

ANALYSIS OF WEAK REGULARITY METHODS FOR SYSTEMS OF PDE USING OSCILLATORY INTEGRALS.

4.1 Outline

We fill in the details of Lemma 2.6 (Step 3) of Chapter 2 by proving Theorem 4.1 through a series of lemmas. First we derive/cite local and global surjectivity results for the divergence operator on the Hilbert space $H^1_0(\Omega, \mathbb{E}^n)$ in Section 4.3. Then we switch to studying a Stokes-like problem, and the rest of the regularity theory takes place in this setting. We prove Lemma 4.4 controlling the pressure locally in terms of the L^2 norm of the gradient of the displacement for solutions to the Stokes-like problem. Then we show the Caccioppoli inequality, Lemma 4.5, which similarly provides local control of the gradient of the displacement. From these estimates it is then convenient to prove higher regularity with respect to the L^2 norm using difference quotients on domains with flat boundary, Lemma 4.6. Then in Lemma 4.8 we show the constant coefficient A-harmonic estimates. Using the technical iteration lemma and freezing coefficients we generalize to variable coefficient estimates, Lemma 4.9. At this point we finish by some powerful tools. The Schauder estimates are shown by the equivalence of the Hölder norm and the norm of Campanato spaces. By bootstrapping the variable coefficient estimates on Morrey spaces then proves the estimates on the Hölder spaces including the BMO space. The Stampacchia interpolation lemma is then used to show the estimates on the L^p Sobolev spaces and prove Theorem 4.1.

4.1.1 Notation

As always the domains $\Omega \subset \mathbb{E}^n$ are open, bounded, and connected.

Definition 4.1. *We say that $\partial\Omega$ is of class $C^{k,\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in (0,1]$ and $k \geq 1$ (or of class $W^{k,p}$ for $k \geq 2$, and $p > n$), if there is a cover of open sets, $\overline{\Omega} \subset \cup_{i=1}^{M} O_i$, and maps $\boldsymbol{\phi}_i \in C^{k,\alpha}(O_i, \mathbb{E}^n)$ (or $\boldsymbol{\phi}_i \in W^{k,p}(O_i, \mathbb{E}^n)$) such that $\boldsymbol{\phi}_i$ is injective on O_i, $\det(\nabla\boldsymbol{\phi}_i(\mathbf{x})) \geq \delta > 0$ for $\mathbf{x} \in O_i$, and there is $\mathbf{x}_i \in O_i$ and a unit vector $\mathbf{n}_i \in \mathbb{E}^n$ such that $(\boldsymbol{\phi}_i(\mathbf{x}) - \boldsymbol{\phi}_i(\mathbf{x}_i)) \cdot \mathbf{n}_i > 0$ if and only if $\mathbf{x} \in \Omega$.*

Remark 4.1. *The case $k = 0$ and $\alpha = 1$ in Definition 4.1 would correspond to the assumption that $\partial\Omega$ is weakly Lipschitz. We always assume the stronger Lipschitz assumption of Definition C.2 that the domain is represented as the graph of a Lipschitz function. With at least C^1 regularity, i.e. the assumptions of Definition 4.1, these two notions are equivalent.*

Many of the manipulations require working on balls for which we define, given $R > 0$, $A \subset \mathbb{E}^n$ open, and $\mathbf{x} \in \overline{A}$,

$$B(\mathbf{x}_0, R, A) = \{\mathbf{x} \in A : |\mathbf{x} - \mathbf{x}_0| < R\},$$

and if the set A is omitted then it is taken to be all of $\mathbb{E}^n$. For measurable sets, $A \subset \mathbb{E}^n$, and $\mathbf{w} \in L^1(A, \mathbb{R}^N)$, we define the mean value as

$$(\mathbf{w})_A = \left(\int_A dV\right)^{-1} \int_A \mathbf{w}\, dV.$$

To avoid excessive sub/super-scripts we use the notation for semi-norms, given $\mathbf{w} \in W^{k,p}(\Omega, \mathbb{R}^N)$

$$\phi^{k,p}(\mathbf{w}, \mathbf{x}_0, R) = \int_{B(\mathbf{x}_0,R,\Omega)} \left|\nabla^k \mathbf{w}\right|^p dV \tag{4.1}$$

$$\psi^{k,p}(\mathbf{w}, \mathbf{x}_0, R) = \int_{B(\mathbf{x}_0,R,\Omega)} \left|\nabla^k \mathbf{w} - (\nabla^k \mathbf{w})_{B(\mathbf{x}_0,R,\Omega)}\right|^p dV. \tag{4.2}$$

Given $\mathbf{f} \in C^1(\overline{\Omega}, \mathbb{E}^3)$, we abbreviate the cofactor matrix as

$$\mathbf{C}(\mathbf{x}) = \operatorname{cof}(\nabla \mathbf{f}(\mathbf{x})).$$

We reference some existence results for the divergence operator on special domains requiring the following definition.

Definition 4.2. *For $A \subset \mathbb{E}^n$ open and bounded, we say that A is* star-shaped *with respect to $B(\mathbf{x}_0, R) \subset A$ if for all $\mathbf{y} \in A$ and $\mathbf{x} \in B(\mathbf{x}_0, R)$ the line connecting $\mathbf{x}$ and $\mathbf{y}$ is contained in A. For such domains, the* star-factor *is $\sigma(A) = R^{-1} \sup_{\mathbf{x}\in A} |\mathbf{x} - \mathbf{x}_0|$.*

We consider linear functionals of $H^1_0(\Omega, \mathbb{E}^3)$ of a special form. Let $\iota : L^2(\Omega, L(\mathbb{E}^3)) \to H^1_0(\Omega, \mathbb{E}^3)$ by

$$\langle \mathbf{v}, \iota[\mathbf{G}] \rangle = \int_\Omega (\nabla \mathbf{v}) \cdot \mathbf{G} \, dV$$

for $\mathbf{v} \in H^1_0(\Omega, \mathbb{E}^3)$.

A final piece of notation is used for the modulus of continuity for functions $w \in C(\overline{\Omega})$ and $\mathbf{x}_0 \in \overline{\Omega}$ and $R > 0$,

$$\omega(w, \mathbf{x}_0, R) = \sup_{\mathbf{y}\in B(\mathbf{x}_0,R,\Omega)} \left|w(\mathbf{y}) - w(\mathbf{x}_0)\right|.$$

4.2 Main Results

The main result needed for Step 3 of Lemma 2.6 is

Theorem 4.1. *Suppose* $\mathbf{f} \in \mathcal{A}_{in}^{W}$ *and* $\partial\Omega$ *is of class* $W^{2,p}$ *for the same* p *value as* $\mathcal{A}_{in}^{W}$*, then the linearized incompressibility operator* $DH[\mathbf{f}] : W^{2,p}(\Omega, \mathbb{E}^3) \cap H_0^1(\Omega, \mathbb{E}^3) \to W^{1,p}(\Omega)$ *has closed range with co-dimension 1 given by the compatibility criterion for* $h \in W^{1,p}(\Omega)$

$$\int_\Omega h\, dV = 0. \tag{4.3}$$

Proof. Given $h \in W^{1,p}(\Omega)$ satisfying (4.3), Proposition 4.1 implies there exists a unique pair $(\mathbf{u}, p) \in H_0^1(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ satisfying $(p)_\Omega = 0$ and

$$\begin{aligned} \int_\Omega (\nabla \mathbf{v}) \cdot [\nabla \mathbf{u} - p\mathbf{C}(\mathbf{x})] dV =& 0\ \forall\ \mathbf{v} \in H_0^1(\Omega, \mathbb{E}^3) \\ \mathbf{C}(\mathbf{x}) \cdot \nabla \mathbf{u}(\mathbf{x}) =& h(\mathbf{x})\ a.e.\ \mathbf{x} \in \Omega. \end{aligned} \tag{4.4}$$

Let $\{O_i\}_{i=1}^M$ and $\boldsymbol{\phi}_i \in W^{2,p}(O_i, \mathbb{E}^3)$ be the sets and maps of Definition 4.1, and let $\tilde{O}_i = \boldsymbol{\phi}_i(O_i \cap \Omega)$ and $\tilde{\Gamma}_i = \boldsymbol{\phi}_i(O_i \cap \partial\Omega)$. We change variables into $\tilde{\mathbf{x}} = \boldsymbol{\phi}_i(\mathbf{x})$ and let $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \mathbf{u}(\boldsymbol{\phi}^{-1}(\tilde{\mathbf{x}}))$ and $\tilde{p}(\tilde{\mathbf{x}}) = p(\boldsymbol{\phi}^{-1}(\tilde{\mathbf{x}}))$. We have $(\tilde{\mathbf{u}}, \tilde{p}) \in H^1(\tilde{O}_i, \mathbb{E}^3) \times L^2(\tilde{O}_i)$, and if $O_i \cap \partial\Omega \neq \emptyset$ then $\tilde{\mathbf{u}}$ vanishes on $\tilde{\Gamma}_i$, the flat part of $\tilde{O}_i$. We find from (4.4) that $(\tilde{\mathbf{u}}, \tilde{p})$ solve

$$\begin{aligned} \int_{\tilde{O}_i} (\tilde{\nabla}\tilde{\mathbf{v}}) \cdot \left[\tilde{\mathbf{A}}(\tilde{\mathbf{x}})\tilde{\nabla}\tilde{\mathbf{u}} - \tilde{p}\tilde{\mathbf{C}}(\tilde{\mathbf{x}})\right] dV =& 0,\ \forall\ \tilde{\mathbf{v}} \in H_0^1(\tilde{O}_i, \mathbb{E}^3), \\ \tilde{\mathbf{C}}(\tilde{\mathbf{x}}) \cdot \tilde{\nabla}\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) =& \tilde{h}(\tilde{\mathbf{x}}),\ a.e.\ \tilde{\mathbf{x}} \in \tilde{O}_i, \end{aligned}$$

for $\tilde{\nabla}$ the gradient with respect to $\tilde{\mathbf{x}}$, and

$$\begin{aligned}
\mathbf{Q}(\tilde{\mathbf{x}}) =& \nabla \boldsymbol{\phi}_i(\boldsymbol{\phi}_i^{-1}(\tilde{\mathbf{x}})), \\
\tilde{\mathbf{A}}(\tilde{\mathbf{x}})\mathbf{B} =& \mathbf{B}\mathbf{Q}(\tilde{\mathbf{x}})\mathbf{Q}^\top(\tilde{\mathbf{x}}) \det\left(\mathbf{Q}(\tilde{\mathbf{x}})\right)^{-1}, \ \forall\ \mathbf{B} \in L(\mathbb{E}^3), \\
\tilde{\mathbf{C}}(\tilde{\mathbf{x}}) =& \mathbf{C}(\boldsymbol{\phi}_i^{-1}(\tilde{\mathbf{x}}))\mathbf{Q}^\top(\tilde{\mathbf{x}}) \det\left(\mathbf{Q}(\tilde{\mathbf{x}})\right)^{-1}, \\
\tilde{h}(\tilde{\mathbf{x}}) =& \det\left(\mathbf{Q}(\tilde{\mathbf{x}})\right)^{-1} h(\boldsymbol{\phi}_i^{-1}(\tilde{\mathbf{x}})).
\end{aligned}$$

From the continuity of the composition operator for $W^{2,p}$ maps, Lemma A.3, and continuity of products of $W^{1,p}$ functions, Lemma A.2, we find that $\tilde{h} \in W^{1,p}(\tilde{O}_i)$, as are the coefficients of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$. Thus from the Sobolev embedding into Hölder spaces, $\tilde{h} \in C^{0,\alpha}(\tilde{O}_i)$ for $\alpha = \frac{p-3}{p} > 0$ along with the coefficients of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$.

We apply Lemma 4.10 to show that for any compact set $K \subset \tilde{O}_i \cup \tilde{\Gamma}_i$, $\tilde{\mathbf{u}} \in C^{1,\alpha}(K, \mathbb{E}^3)$ and $\tilde{p} \in C^{0,\alpha}(K)$, i.e. regularity holds up to the flat boundary. Next we apply the second half of Lemma 4.6 to show that $\tilde{\mathbf{u}} \in H^2(K, \mathbb{E}^3)$. Consider $\mathbf{j} \in \mathbb{E}^3$ such that $\mathbf{j} \cdot \mathbf{n}_i = 0$ for $\mathbf{n}_i$ normal to the flattened boundary (as in Definition 4.1). Let $\tilde{\mathbf{u}}_j = (\tilde{\nabla}\tilde{\mathbf{u}})\mathbf{j} - \tilde{\mathbf{u}}_0$ and $\tilde{p}_j = (\tilde{\nabla}\tilde{p}) \cdot \mathbf{j} - ((\tilde{\nabla}\tilde{p}) \cdot \mathbf{j})_{\tilde{O}_i}$, where $\tilde{\mathbf{u}}_0$ is smooth on K, vanishes on $\tilde{\Gamma}_i$, and agrees with $(\tilde{\nabla}\tilde{\mathbf{u}})\mathbf{j}$ on $\partial\tilde{O}_i \backslash \tilde{\Gamma}_i$. Then $(\tilde{\mathbf{u}}_j, \tilde{p}_j) \in H_0^1(\tilde{O}_i, \mathbb{E}^3) \times L^2(\tilde{O}_i)$ solve

$$\begin{aligned}
\int_{\tilde{O}_i} (\tilde{\nabla}\tilde{\mathbf{v}}) \cdot \left[\tilde{\mathbf{A}}(\tilde{\mathbf{x}})\tilde{\nabla}\tilde{\mathbf{u}}_j - \tilde{p}_j\tilde{\mathbf{C}}(\tilde{\mathbf{x}}) - \tilde{\mathbf{G}}_j\right] dV =& 0, \ \forall\ \tilde{\mathbf{v}} \in H_0^1(\tilde{O}_i, \mathbb{E}^3), \\
\tilde{\mathbf{C}}(\tilde{\mathbf{x}}) \cdot \tilde{\nabla}\tilde{\mathbf{u}}_j(\tilde{\mathbf{x}}) =& \tilde{h}_j(\tilde{\mathbf{x}}), \ a.e.\ \tilde{\mathbf{x}} \in \tilde{O}_i,
\end{aligned} \tag{4.5}$$

for $\tilde{\partial}_j$ the directional derivative with respect to $\mathbf{j}$ and

$$\begin{aligned}
\tilde{\mathbf{G}}_j =& -(\tilde{\partial}_j\tilde{\mathbf{A}}(\tilde{\mathbf{x}}))\tilde{\nabla}\tilde{\mathbf{u}} - \tilde{\mathbf{A}}(\tilde{\mathbf{x}})\tilde{\nabla}\tilde{\mathbf{u}}_0 + \tilde{p}_j\tilde{\partial}_j\tilde{\mathbf{C}}(\tilde{\mathbf{x}}) \\
\tilde{h}_j(\mathbf{x}) =& \tilde{\partial}_j\tilde{h} - (\tilde{\partial}_j\tilde{\mathbf{C}}(\tilde{\mathbf{x}}))\tilde{\nabla}\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) - \tilde{\mathbf{C}}(\tilde{\mathbf{x}})\tilde{\nabla}\tilde{\mathbf{u}}_0(\tilde{\mathbf{x}}).
\end{aligned}$$

By Proposition 4.1 the solution operator to (4.5) is bounded from $L^2(\tilde{O}_i, L(\mathbb{E}^3)) \times L^2(\tilde{O}_i)/\mathbb{R}$ into $H_0^1(\tilde{O}_i, \mathbb{E}^3) \times L^2(\tilde{O}_i)\backslash\mathbb{R}$. Thus let $T_K[\tilde{\mathbf{G}}_j, \tilde{h}_j] = (\tilde{\nabla}\tilde{\mathbf{u}}_j, \tilde{p}_j)$, and T map $L^2(\tilde{O}_i, L(\mathbb{E}^3)) \times L^2(\tilde{O}_i)$ into $L^2(K, L(\mathbb{E}^3)) \times L^2(K)$ by taking the gradient of the displacement and restricting to the compact set K. Lemma 4.10 shows that T_K is bounded mapping $L^\infty(\tilde{O}_i, L(\mathbb{E}^3)) \times L^\infty(\tilde{O}_i)$ into $BMO(K, L(\mathbb{E}^3)) \times BMO(K)$ by the embedding of $L^\infty(\tilde{O}_i)$ into $\mathcal{L}^{p,n}(\tilde{O}_i)$ and the equivalence of $\mathcal{L}^{p,n}(K)$ and $BMO(K)$. Thus the interpolation Theorem 4.2 shows that T_K is bounded mapping the L^p spaces into L^p spaces. We can easily check that $(\tilde{\mathbf{G}}_j, \tilde{h}_k) \in L^p(\tilde{O}_i, L(\mathbb{E}^3)) \times L^p(\tilde{O}_i)$ thus for the tangential direction $\mathbf{j}$, $(\tilde{\nabla}\tilde{\mathbf{u}}_j, \tilde{p}_j) \in L^p(K, L(\mathbb{E}^3)) \times L^p(K)$. In Lemma 4.7 we solve for the normal derivatives in the strong form of the equation, showing that indeed $\tilde{\mathbf{u}} \in W^{2,p}(K, \mathbb{E}^3)$. Finally, we choose the compact sets $K_i \subset O_i \cap \overline{\Omega}$ such that $\overline{\Omega} \subset \cup_{i=1}^M \boldsymbol{\phi}_i^{-1}(K_i)$, and since $\tilde{\mathbf{u}} \in W^{2,p}(K_i, \mathbb{E}^3)$ the continuity of the composition implies that $\mathbf{u} \in W^{2,p}(\Omega, \mathbb{E}^3)$. $\square$

4.3 Linear Existence Theory

For a basic existence lemma we use the method from [25]. However, later we need a stronger existence result.

Lemma 4.1. *Suppose $n = 3$ and $h \in L^2(B(\mathbf{0}, R))$ and $(h)_{B(\mathbf{0},R)} = 0$, then there exists $\mathbf{u} \in H_0^1(B(\mathbf{0}, R))$ such that $\nabla \cdot \mathbf{u}(\mathbf{x}) = h(\mathbf{x})$ for almost every $\mathbf{x} \in B(\mathbf{0}, R)$ and using the notation of (4.1),*

$$\phi^{1,2}(\mathbf{u}, \mathbf{0}, R) \leq C\phi^{0,2}(h, \mathbf{0}, R), \tag{4.6}$$

where the constant is independent of R and h.

Proof. First we change variables into $B(\mathbf{0}, 1)$ in order to remove the dependence on the radius. Let $\tilde{\mathbf{x}} = R^{-1}\mathbf{x}$ and let $\tilde{h}(\tilde{\mathbf{x}}) = h(R\tilde{\mathbf{x}})$. Then we solve the Poisson problem

$$\tilde{\Delta} w(\tilde{\mathbf{x}}) = \tilde{h}(\tilde{\mathbf{x}}), \ \tilde{\mathbf{x}} \in B,$$
$$\tilde{\mathbf{x}} \cdot \tilde{\nabla} w = 0, \ \tilde{\mathbf{x}} \in \partial B.$$

The solution satisfies $w \in H^2(B)$ [13] and

$$\|w\|^2_{H^2(B)} \leq C\phi^{0,2}(h, \mathbf{0}, 1).$$

We next construct a vector field $\mathbf{d} \in H^2(B, \mathbb{E}^3)$ such that $\nabla \times \mathbf{d} = \nabla w$ for $\tilde{\mathbf{x}} \in \partial B$. Let

$$(\tilde{\nabla}\mathbf{d}(\tilde{\mathbf{x}}))\tilde{\mathbf{x}} = \tilde{\mathbf{x}} \times \tilde{\nabla} w(\tilde{\mathbf{x}})$$

for $\tilde{\mathbf{x}} \in \partial B$ and extend $\mathbf{d}$ to B such that $\|\mathbf{d}\|_{H^2(B,\mathbb{E}^3)} \leq C\|w\|_{H^2(B)}$. Since $\tilde{\nabla}\mathbf{d}$ is non-zero except for the the normal derivatives on ∂B we compute that for $\tilde{\mathbf{x}} \in \partial B$,

$$\begin{aligned}\tilde{\nabla} \times \mathbf{d}(\tilde{\mathbf{x}}) &= \tilde{\mathbf{x}} \times (\tilde{\nabla}\mathbf{d}(\tilde{\mathbf{x}}))\tilde{\mathbf{x}} \\ &= \tilde{\mathbf{x}} \times (\tilde{\mathbf{x}} \times \tilde{\nabla} w(\tilde{\mathbf{x}})) \\ &= -\tilde{\nabla} w(\tilde{\mathbf{x}}).\end{aligned}$$

Thus let $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \tilde{\nabla} w(\tilde{\mathbf{x}}) + \tilde{\nabla} \times \mathbf{d}(\tilde{\mathbf{x}})$ and $\tilde{\mathbf{u}} \in H^1_0(B, \mathbb{E}^3)$ such that $\tilde{\nabla} \cdot \tilde{\mathbf{u}} = \tilde{h}$ and there is a constant such that

$$\phi^{1,2}(\tilde{\mathbf{u}}, \mathbf{0}, 1) \leq C\phi^{0,2}(\tilde{h}, \mathbf{0}, 1).$$

Changing variables let $\mathbf{u}(\mathbf{x}) = R\tilde{\mathbf{u}}(R^{-1}\mathbf{x})$ so that $\nabla \cdot \mathbf{u}(\mathbf{x}) = h(\mathbf{x})$ and

$$\begin{aligned}\phi^{1,2}(\mathbf{u}, \mathbf{0}, R) =& \int_B |\tilde{\nabla}\tilde{\mathbf{u}}|^2 R^3 dV \\ \leq& C \int_B |\tilde{h}|^2 R^3 dV \\ \leq& C\phi^{0,2}(h, \mathbf{0}, R).\end{aligned}$$

□

This proof can easily be adapted for showing the Hodge decomposition in higher dimensions and more general domains with for example C^2 boundary. Given data with higher regularity, we could construct solutions with higher regularity. However, due to the reliance on the regularity of the potential functions, i.e. w and $\mathbf{d}$, we will always require one more derivative of regularity of the boundary than we achieve for the vector field $\mathbf{u}$. In Theorem 4.1, we require the vector field to be of class $W^{2,p}$ and this would require at least $W^{3,p}$ maps for the boundary. However, since the deformation $\mathbf{f}$ is of class $W^{2,p}$ we can not smooth out the boundary any more and still change to coordinates for which the linearized incompressibility operator is the divergence operator.

We consider an alternate approach attributed to Bogovskii [8] and presented in detail by Galdi [14]. This can be applied to domains with strongly Lipschitz boundary, and we use this in Lemma 2.5 and Step 4 of Lemma 2.6, as well as in this chapter.

Lemma 4.2. *Suppose that Ω is star-shaped with respect to $B(\mathbf{x}_0, R)$ and $h \in W_0^{k,p}(\Omega)$ such that $\int_\Omega h\, dV = 0$ for $k \geq 0$ and $p \in (1, \infty)$. Then there is $\mathbf{u} \in W_0^{k+1,p}(\Omega, \mathbb{E}^n)$ with $\nabla \cdot \mathbf{u} = h$,* *and there is a constant, depending only on $\sigma(\Omega), k, p, n$ such that*

$$\int_{\Omega} |\nabla^{k+1}\mathbf{u}|^p dV \leq C(\sigma(\Omega), k, p, n) \int_{\Omega} |\nabla^k h|^p dV. \tag{4.7}$$

We do not present the proof as it is rather technical. The method of Bogovskii is to use an explicit integral representation:

$$\mathbf{u}(\mathbf{x}) = \int_{\Omega} h(\mathbf{y}) \left[\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} \int_{|\mathbf{x}-\mathbf{y}|}^{\infty} \omega\left(\mathbf{x}_0 + \mathbf{y} + \xi \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right) \xi^{n-1} d\xi \right] dV_y, \tag{4.8}$$

where $\omega \in C^\infty(B(\mathbf{x}_0, R))$ is non-negative, integrates to 1, and has compact support.

The solution has support in the convex hull of the support of h and $B(\mathbf{x}_0, R)$. From there, it is shown that any Lipschitz domain can be expressed as the union of finitely many of such star-shaped domains and the data can be decomposed suitably. This result is useful in Chapter 2 for the case of strong Dirichlet boundary conditions. However, for weak Dirichlet boundary conditions it is not directly useful. If the integral representation is applied to data in $W^{1,p}(\Omega)$ which does not vanish on the boundary, then it is clear that we obtain a displacement $\mathbf{u} \in W_0^{1,p}(\Omega, \mathbb{E}^n)$, but $\mathbf{u}$ may not have higher regularity at the boundary.

Therefore we focus on the approach of Giaquinta and Modica [17], with which they show regularity for solutions to Stokes-like systems. The methods are important for the study of regularity of weak solutions for linearized elasticity equations and are used in [17] to study regularity for some non-linear equations, although we do not attempt here for any such general results.

The equations we consider may be expressed abstractly as an operator $S_{A,f}$:

$H^1(\Omega, \mathbb{E}^n) \times L^2(\Omega) \rightarrow H_0^1(\Omega, \mathbb{E}^n)^* \times L^2(\Omega)$. We consider the case that $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$ for $\mathbf{G} \in L^2(\Omega, L(\mathbb{E}^3))$, which is equivalent to

$$\begin{aligned} \int_\Omega (\nabla \mathbf{v}) \cdot [\mathbf{A}(\mathbf{x})\nabla \mathbf{u} - p\mathbf{C}(\mathbf{x}) - \mathbf{G}] dV &= 0, \ \forall \ \mathbf{v} \in H_0^1(\Omega, \mathbb{E}^3), \\ \mathbf{C}(\mathbf{x}) \cdot \nabla \mathbf{u}(\mathbf{x}) &= h(\mathbf{x}), \ \ a.e. \ \mathbf{x} \in \Omega. \end{aligned} \tag{4.9}$$

We may consider more general $\mathbf{A} \in L(L(\mathbb{E}^n))$ that satisfy the following: there exists $\lambda > 0$ and $R_0 > 0$ such that for $\mathbf{x}_0 \in \overline{\Omega}$ and $0 < R < R_0$,

$$\lambda \phi^{1,2}(\mathbf{v}, \mathbf{x}_0, R, \Omega) \leq \int_{B(\mathbf{x}_0, R, \Omega)} (\nabla \mathbf{v}) \cdot \mathbf{A}(\mathbf{x}) \nabla \mathbf{v} \, dV, \ \forall \ \mathbf{v} \in H_0^1(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^n). \tag{4.10}$$

This is satisfied if $\mathbf{A}(\mathbf{x})$ is uniformly positive definite on $L(\mathbb{E}^n)$, but in fact is satisfied in other cases as well. For example for linear elasticity, (4.10) is verified by Korn's lemma. This condition generalizes to non-linear operators as strict quasi-convexity.

We use a related property of the linearized incompressibility constraint.

Lemma 4.3. *Suppose $\mathbf{f} \in \mathcal{A}_{in}^W$ and $\partial\Omega$ is strongly Lipschitz, there exists $\lambda > 0$ and $R_0 > 0$ such that for $\mathbf{x}_0 \in \overline{\Omega}$ and $0 < R < R_0$ and $h \in L^2(B(\mathbf{x}_0, R, \Omega))$ with $\int_{B(\mathbf{x}_0,R,\Omega)} h \, dV = 0$, then there is $\mathbf{u} \in H_0^1(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$ such that $\mathbf{C}(\mathbf{x}) \cdot \nabla \mathbf{u}(\mathbf{x}) = h(\mathbf{x})$ a.e. $\mathbf{x} \in \Omega$ and*

$$\lambda \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) \leq \phi^{0,2}(h, \mathbf{x}_0, R).$$

If furthermore, $h \in W_0^{1,p}(B(\mathbf{x}_0, R, \Omega))$ then $\mathbf{u} \in W_0^{2,p}(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$.

Proof. Since $\nabla \mathbf{f}$ is uniformly continuous and non-singular, we may choose R_0 such that $\mathbf{f}$ is injective on $\overline{B(\mathbf{x}_0, R, \Omega)}$ for all $\mathbf{x}_0 \in \overline{\Omega}$ and $0 < R < R_0$, and $\mathbf{f}(B(\mathbf{x}_0, R, \Omega))$ is star-shaped with star-factor less than 4 for all $0 < R < R_1$ and $\mathbf{x}_0 \in \overline{\Omega}$. Then change variables to the ball in the current configuration and applying Lemma 4.2 with $\tilde{h} = \det(\nabla \mathbf{f})^{-1} h$ implies the existence of $\tilde{\mathbf{u}} \in H_0^1(\mathbf{f}(B(\mathbf{x}_0, R, \Omega)), \mathbb{E}^3)$ such that $\tilde{\nabla} \cdot \tilde{\mathbf{u}} = \tilde{h}$. Changing coordinates back we find

$$\begin{aligned}\mathbf{C}(\mathbf{x}) \cdot \nabla \mathbf{u}(\mathbf{x}) =& \mathbf{C}(\mathbf{x}) \cdot \tilde{\nabla} \tilde{\mathbf{u}}(\mathbf{f}(\mathbf{x})) \nabla \mathbf{f}(\mathbf{x}) \\ =& \operatorname{cof}(\nabla \mathbf{f}(\mathbf{x})) \nabla \mathbf{f}(\mathbf{x})^{\top} \cdot \tilde{\nabla} \tilde{\mathbf{u}}(\mathbf{f}(\mathbf{x})) \\ =& \det(\nabla \mathbf{f}(\mathbf{x})) \tilde{h}(\mathbf{f}(\mathbf{x})) \\ =& h(\mathbf{x}).\end{aligned}$$

The uniform estimate follows from the uniform bound on the star-factor. □

Corollary 4.1. *Given $\mathbf{f} \in \mathcal{A}_{in}^W$ and $h \in L^2(\Omega)$ such that $\int_\Omega h \, dV = 0$, there exists $\mathbf{u} \in H_0^1(\Omega, \mathbb{E}^3)$ such that $\mathbf{C} \cdot \nabla \mathbf{u} = h$ and there is a constant $\lambda > 0$ independent of h such that*

$$\lambda \int_\Omega |\nabla \mathbf{u}|^2 dV \leq \int_\Omega |h|^2 dV.$$

In the case that $h \in W_0^{1,p}(\Omega, \mathbb{E}^3)$, the displacement satisfies $\mathbf{u} \in W_0^{2,p}(\Omega, \mathbb{E}^3)$.

Proof. We cover $\overline{\Omega}$ by finitely many balls, $\{B_i\}_{i=1}^M$, of radius less than R_0 from Lemma 4.3. We iteratively decompose h into h_i with compact support in B_i, $(h_i)_{B_i} = 0$ for each i and still $\sum_{i=1}^M h_i(\mathbf{x}) = h(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$. It is shown in Lemma 3.2 [14] how to do this procedure and achieve $\|h_i\|_{L^2(B_i)}^2 \leq C\|h\|_{L^2(\Omega)}^2$ with the constant independent of h. Then Lemma 4.3 constructs $\mathbf{u}_i \in H_0^1(B_i, \mathbb{E}^n)$, such that $\mathbf{C}(\mathbf{x}) \cdot \nabla \mathbf{u}_i(\mathbf{x}) = h_i(\mathbf{x})$ a.e. $\mathbf{x} \in B_i$, and $\mathbf{u} = \sum_{i=1}^M \mathbf{u}_i$ is the desired displacement. □

Proposition 4.1. *Given $\mathbf{f} \in \mathcal{A}_{in}^{W}$, we consider the Stokes-like system with $\mathbf{A}(\mathbf{x})\mathbf{B} = \mathbf{B}$ for all $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{B} \in L(\mathbb{E}^3)$. Let $\mathbf{G} \in L^2(\Omega, L(\mathbb{E}^3))$ and $h \in L^2(\Omega)$ with $(h)_\Omega = 0$, then there exists $\mathbf{u} \in H_0^1(\Omega, \mathbb{E}^3)$ and $p \in L^2(\Omega)$ such that $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$. Furthermore, $(\mathbf{u}, p)$ are unique if we set $(p)_\Omega = 0$.*

Proof. We consider the linearized constraint as a linear operator on the Hilbert spaces, $DH[\mathbf{f}] : H_0^1(\Omega, \mathbb{E}^3) \to L^2(\Omega)$. Lemma 4.1 implies there is $\mathbf{u}_1 \in H_0^1(\Omega, \mathbb{E}^3)$ such that $DH[\mathbf{f}]\mathbf{u}_1 = h$. By continuity of the linearized constraint on the Hilbert space, $\ker(DH[\mathbf{f}]) \subset H_0^1(\Omega, \mathbb{E}^3)$ is a closed sub-space and a Hilbert space with inner product

$$\int_\Omega (\nabla \mathbf{v}) \cdot \nabla \mathbf{u} \, dV.$$

The Reisz representation theorem implies that $\iota[\mathbf{G} - \nabla \mathbf{u}_1]$ may be represented by $\mathbf{u}_2 \in \ker(DH[\mathbf{f}])$ such that, for $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$,

$$\int_\Omega (\nabla \mathbf{v}) \cdot [\nabla \mathbf{u} - \mathbf{G}] \, dV = 0 \; \forall \; \mathbf{v} \in \ker(DH[\mathbf{f}]).$$

Since $DH[\mathbf{f}]$ has closed range by Lemma 4.1, the closed range theorem implies that $\iota[\nabla \mathbf{u} - \mathbf{G}] \in \ker(DH[\mathbf{f}])^\perp = R(DH[\mathbf{f}]^*)$ so $\iota[\nabla \mathbf{u} - \mathbf{G}] = DH[\mathbf{f}]^* p$ for $p \in L^2(\Omega)$, and $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$. To show uniqueness, suppose $(\mathbf{u}_\alpha, p_\alpha)$ are solutions with $\alpha \in \{1, 2\}$ and $(p_\alpha)_\Omega = 0$. Then let $(\mathbf{w}, q) = (\mathbf{u}_1 - \mathbf{u}_2, p_1 - p_2)$ and since $\mathbf{w} \in \ker(DH[\mathbf{f}])$,

$$0 = \int_\Omega |\nabla \mathbf{w}|^2 dV, \tag{4.11}$$

which shows that $\mathbf{w}$ is constant. Because $\mathbf{w} \in H_0^1(\Omega, \mathbb{E}^3)$, it is identically 0. Then

$$\int_\Omega (\nabla \mathbf{v}) \cdot \mathbf{C} q \, dV = 0 \; \forall \; \mathbf{v} \in H_0^1(\Omega, \mathbb{E}^3).$$

so since $(q)_\Omega = 0$, by Corollary 4.1 we may choose $\mathbf{v}$ such that $DH[\mathbf{f}]\mathbf{v} = q$ and thus $\int_\Omega |q|^2 dV = 0$ so $q = 0$. □

This completes the existence arguments. To prove Theorem 4.1 we show that given higher regularity of $\partial\Omega$, h and $\mathbf{f}$, the displacement $\mathbf{u}$ found in Proposition 4.1 is in fact in $W^{2,p}(\Omega, \mathbb{E}^3)$.

4.4 Linear Regularity Theory

4.4.1 Localization Lemmas

The Caccioppoli inequality, Lemma 4.5, and 'pressure control', Lemma 4.4, involve two basic but important localization arguments.

Lemma 4.4 (Pressure Control). *Suppose $\partial\Omega$ is Lipschitz and $\mathbf{f} \in \mathcal{A}^W_{in}$. Let $(\mathbf{u}, p) \in H^1(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ solve $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$ with $\mathbf{G} \in L^2(\Omega, L(\mathbb{E}^3))$ and $h \in L^2(\Omega)$. Then there exists a constant C and $R_0 > 0$ (dependent only on $\Omega, \mathbf{f}, |\mathbf{A}|$) such that for all $\mathbf{x}_0 \in \overline{\Omega}$ and $0 < R < R_0$,*

$$\psi^{0,2}(p, \mathbf{x}_0, R) \le C\left(\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + \phi^{0,2}(\mathbf{G}, \mathbf{x}_0, R)\right). \tag{4.12}$$

Proof. We choose R_0 from Lemma 4.3. Then we select $\mathbf{v} \in H^1_0(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$ such that $\mathbf{C} \cdot \nabla \mathbf{v} = p - (p)_{B(\mathbf{x}_0,R,\Omega)}$. From (4.9) we find that

$$\begin{aligned}\psi^{0,2}(p, \mathbf{x}_0, R) &= \int_{B(\mathbf{x}_0,R,\Omega)} (\nabla \mathbf{v}) \cdot [\mathbf{A}(\mathbf{x})\nabla \mathbf{u} - \mathbf{G}]\, dV \\ &\leq \left(\lambda^{-1}\psi^{0,2}(p, \mathbf{x}_0, R)\right)^{\frac{1}{2}} \left(1 + \sup_{\mathbf{x}\in\overline{\Omega}} |\mathbf{A}(\mathbf{x})|\right) \left(\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + \phi^{0,2}(\mathbf{G}, \mathbf{x}_0, R)\right)^{\frac{1}{2}}\end{aligned}$$

where λ if from Lemma 4.3 and we used the Cauchy-Schwartz inequality. This shows (4.12) with $C = \lambda^{-1}\left(1 + \sup_{\mathbf{x}\in\overline{\Omega}} |\mathbf{A}(\mathbf{x})|\right)^2$. □

Next we control the term $\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R)$.

Lemma 4.5 (Caccioppoli Inequality)**.** *Suppose $\partial\Omega$ is Lipschitz and $\mathbf{f} \in \mathcal{A}^W_{in}$. Suppose also that $\mathbf{A}$ satisfies the local coercivity condition, (4.10). Let $(\mathbf{u}, p) \in H^1(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ solve $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$ with $\mathbf{G} \in L^2(\Omega, L(\mathbb{E}^3))$ and $h \in L^2(\Omega)$.*

Then there is some $R_0 > 0$ and a constant C such that if $0 < R < R_0$ and $\mathbf{u}$ vanishes on $\partial\Omega \cap B(\mathbf{x}_0, R)$, then

$$\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) \leq C\left(\frac{1}{R^2}\phi^{0,2}(\mathbf{u}, \mathbf{x}_0, 2R) + \phi^{0,2}(\mathbf{G}, \mathbf{x}_0, 2R) + \phi^{0,2}(h, \mathbf{x}_0, 2R)\right). \tag{4.13}$$

Proof. Choose $R_0 = \frac{1}{2}\min\{R_1, R_2\}$ for R_1 the corresponding parameter from Lemma 4.4 and R_2 corresponding to the local coercivity condition (4.10). We use a smooth bump function η such $|\nabla\eta| \leq \frac{C_1}{R}$, $\eta(\mathbf{x}) = 1$ for $\mathbf{x} \in B(\mathbf{x}_0, R, \Omega)$ and $\eta \geq 0$ and compactly supported in $B(\mathbf{x}_0, 2R, \Omega)$. Let $\mathbf{v} = \eta^2\mathbf{u} \in H^1_0(B(\mathbf{x}_0, 2R, \Omega), \mathbb{E}^n)$ and use coercivity to get that $\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R)$ is bounded by

$$\begin{aligned}\int_{B(\mathbf{x}_0,2R,\Omega)} \left|\nabla(\eta\mathbf{u})\right|^2 dV &\leq \int_{B(\mathbf{x}_0,2R,\Omega)} (\nabla(\eta\mathbf{u})) \cdot \mathbf{A}(\mathbf{x})\nabla(\eta\mathbf{u}) dV \\ &\leq \int_\Omega \Big[(\nabla\mathbf{v}) \cdot \mathbf{A}(\mathbf{x})\nabla\mathbf{u} + 2|\mathbf{A}(\mathbf{x})|\, |\nabla\eta|\, |\nabla\mathbf{u}|\, |\mathbf{u}|\Big] dV \\ &\leq \int_\Omega (\nabla\mathbf{v}) \cdot \mathbf{A}(\mathbf{x})\nabla\mathbf{u}\, dV + \frac{C_2(\epsilon)}{R^2}\phi^{0,1}(\mathbf{u}, \mathbf{x}_0, 2R) + \frac{\epsilon}{2}\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, 2R)\end{aligned}$$

using the standard trick that $|ab| \leq C(\epsilon)a^2 + \epsilon b^2$ for arbitrary $\epsilon > 0$ where $a = \frac{2C_1}{R}\|\mathbf{A}(\mathbf{x})\|\, |\mathbf{u}|$ and $b = |\nabla\mathbf{u}|$.

From the first term we use the equation, (4.9), and must deal with the term

$$\int_\Omega (\nabla\mathbf{v}) \cdot p\mathbf{C}(\mathbf{x}) dV = \int_{B(\mathbf{x}_0,2R,\Omega)} (p - (p)_{B(\mathbf{x}_0,2R,\Omega)}) \left(\eta^2 h + 2\eta\mathbf{C}(\mathbf{x}) \cdot (\mathbf{u} \otimes \nabla\eta)\right) dV, \qquad (4.14)$$

and the mean value of p was inserted because $\nabla \cdot \mathbf{C}(\mathbf{x}) = 0$ and $\mathbf{v} \in H_0^1(B(\mathbf{x}_0, 2R, \Omega), \mathbb{E}^n)$ thus any constant term vanishes.

From Lemma 4.4 and repeated applications of the Cauchy-Shwartz inequality and ϵ trick we find that $\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R)$ is now bounded by

$$C(\epsilon)\left(\frac{1}{R^2}\phi^{0,2}(\mathbf{u}, \mathbf{x}_0, 2R) + \phi^{0,2}(\mathbf{G}, \mathbf{x}_0, 2R) + \phi^{0,2}(h, \mathbf{x}_0, 2R)\right) + \epsilon\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, 2R)$$

and the result follows from the ϵ-iteration of Lemma 4.12. □

4.4.2 Higher Regularity

Two methods for higher regularity are mollification and showing the limit converges in a higher regularity norm, or difference quotients. We prefer to use difference quotients for working up to the boundary. The precise form of the the righthand side in (4.18) is important for later estimates. We consider domains of the type used in the proof of Theorem 4.1, where either $\mathbf{x}_1 \in \Omega$ and $\Gamma = \emptyset$ or $\partial\Omega$ contains a closed set Γ such that $\mathbf{x}_1 \in \Gamma$ and there is a unit vector $\mathbf{n}$ such that $(\mathbf{x} - \mathbf{x}_1) \cdot \mathbf{n} = 0$ for all $\mathbf{x} \in \Gamma$. In either case we prove the estimates on balls with $\overline{B} \cap \partial\Omega = \overline{B} \cap \Gamma$. Denote by $H^1_\Gamma(\Omega, \mathbb{E}^3)$ the H^1 closure of displacements that vanish on Γ.

Lemma 4.6 (Higher L^2 regularity)**.** *Assume Ω and Γ are as above. Let $(\mathbf{u}, p) \in H^1_\Gamma(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ solve $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$. Consider two cases,*

1. *$\mathbf{A}$ is uniformly symmetric positive definite with constant coefficients and $\mathbf{C}$ is invertible and constant. Let $\mathbf{G} \in H^{k+1}(\Omega, L(\mathbb{E}^n))$ and $h \in H^{k+1}(\Omega)$ for integer $k \geq 0$. Then for any compact set $K \subset \Omega \cup \Gamma$, $\mathbf{u} \in H^{k+2}(K, \mathbb{E}^n)$ and $p \in H^{k+1}(K)$ and there exists some $C(\delta) > 0$ depending only on $\delta > 0$ and k, and the coefficients, such that for $\mathbf{x}_0 \in \Omega \cup \Gamma$ and $R > 0$ such that $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) > R + \delta$ then*

$$\phi^{k+2,2}(\mathbf{u}, \mathbf{x}_0, R/2) + \phi^{k+1,2}(p, \mathbf{x}_0, R/2) \leq \\ C\left(\frac{1}{R^2}\phi^{k+1}(\mathbf{u}, \mathbf{x}_0, R) + \phi^{k+1}(\mathbf{G}, \mathbf{x}_0, R) + \phi^{k+1}(h, \mathbf{x}_0, R)\right). \tag{4.15}$$

2. *The coefficients of $\mathbf{A}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$ are in $W^{1,p}(\Omega)$ and $\mathbf{A}$ is uniformly symmetric positive definite and $\mathbf{C}$ satisfies the conditions of Lemma 4.3. Then suppose that $(\mathbf{u}, p) \in C^1(\overline{\Omega}, \mathbb{E}^n) \times C(\overline{\Omega})$ and $\mathbf{G} \in H^1(\Omega, L(\mathbb{E}^n))$ and $h \in H^1(\Omega)$. Then for compact $K \subset \Omega \cup \Gamma$, $\mathbf{u} \in H^2(K, \mathbb{E}^n)$ and $p \in H^1(K)$ and there exists some $C(\delta, D) > 0$ depending only on $\delta > 0$, $D > \|\nabla\mathbf{u}\|_\infty + \|p\|_\infty$, and the coefficients, such that for $\mathbf{x}_0 \in \Omega \cup \Gamma$ and $R > 0$ such that $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) > R + \delta$ then (4.18) holds with $k = 0$.*

Proof. Most of the steps may be done for either case. Pick a unit vector $\mathbf{j}$ tangential to the boundary, i.e. $\mathbf{j} \cdot \mathbf{n} = 0$ or any vector if $\Gamma = \emptyset$, let $D_h^j \mathbf{v}(\mathbf{x}) = \frac{1}{h}(\mathbf{v}(\mathbf{x} + h\mathbf{j}) - \mathbf{v}(\mathbf{x}))$. Then for $K \subset \Omega \cup \Gamma$ with $d(K, \partial\Omega \backslash \Gamma) > \delta$, for $|h| \leq \delta$ and $\mathbf{v} \in H_0^1(K, \mathbb{E}^3)$ extended by zero to Ω,

$$\begin{aligned} 0 &= \int_\Omega \left(D_{-h}^j \nabla \mathbf{v}\right) \cdot [\mathbf{A}(\mathbf{x})\nabla\mathbf{u} - p\mathbf{C}(\mathbf{x}) - \mathbf{G}]\, dV \\ &= \int_\Omega (\nabla \mathbf{v}) \cdot \left[\mathbf{A}(\mathbf{x}) D_h^j \nabla \mathbf{u} - (D_h^j p)\mathbf{C}(\mathbf{x}) - \hat{\mathbf{G}}\right] dV \end{aligned} \tag{4.16}$$

and

$$\mathbf{C}(\mathbf{x}) \cdot \nabla D_h^j \mathbf{u} = \hat{h} \tag{4.17}$$

by integration by parts and the product rule for difference quotients, where we have let

$$\begin{aligned} \hat{\mathbf{G}} &= D_h^j \mathbf{G} - (D_h^j \mathbf{A}(\mathbf{x}))\nabla\mathbf{u} + p D_h^j \mathbf{C}(\mathbf{x}), \\ \hat{h} &= D_h^j h - (D_h^j \mathbf{C}(\mathbf{x}))\nabla\mathbf{u}. \end{aligned}$$

Also $D_h^j \mathbf{u}$ vanishes for $\mathbf{x} \in \Gamma \cap K$ so from Lemmas 4.4, 4.5 there are uniform bounds on the norms for $(D_h^j \mathbf{u}, D_h^j p)$ and the difference quotient machinery, see [13], allows us to get that $(\nabla\mathbf{u})\mathbf{j} \in H^1(K, \mathbb{E}^n)$ and $\mathbf{j} \cdot \nabla p \in L^2(K)$. In the first case, repeating this argument we get $((\nabla^l \mathbf{u})(\mathbf{j}_1, \ldots, \mathbf{j}_l), (\nabla^l p)(\mathbf{j}_1, \ldots, \mathbf{j}_l)) \in H^1(K, \mathbb{E}^n) \times L^2(K)$ for $l \leq k$ and $\mathbf{j}_i \cdot \mathbf{n} = 0$ for each $i \in \{1, \ldots, l\}$. Furthermore, from the lemmas, the inequality of the form (4.18) holds for these derivatives. This shows the regularity in the interior of Ω but we must argue that regularity holds for the normal derivatives up to Γ. This argument has applications later so is finished in Lemma 4.7. □

Lemma 4.7. *Suppose* $(\mathbf{u}, p) \in H^2(\Omega, \mathbb{E}^3) \times H^1(\Omega)$ *solve* $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$. *Consider three cases,*

1. $\mathbf{A}$ *is uniformly symmetric positive definite with constant coefficients and* $\mathbf{C}$ *is invertible and constant. Let* $\mathbf{G} \in H^{k+1}(\Omega, L(\mathbb{E}^n))$ *and* $h \in H^{k+1}(\Omega)$ *for integer* $k \geq 0$ *and that* $((\nabla^l \mathbf{u})(\mathbf{j}_1, \ldots, \mathbf{j}_l), (\nabla^l p)(\mathbf{j}_1, \ldots, \mathbf{j}_l)) \in H^1(K, \mathbb{E}^3) \times L^2(K)$ *for* $l \leq k$ *and* $\mathbf{j}_i \cdot \mathbf{n} = 0$ *for each* $i \in \{1, \ldots, l\}$.. *Then for any compact set* $K \subset \Omega \cup \Gamma$, $\mathbf{u} \in H^{k+2}(K, \mathbb{E}^3)$ *and* $p \in H^{k+1}(K)$ *and there exists some* $C(\delta) > 0$ *depending only on* $\delta > 0$ *and* k, *and the coefficients, such that for* $\mathbf{x}_0 \in \Omega \cup \Gamma$ *and* $R > 0$ *such that* $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) > R + \delta$ *then*
$$\phi^{k+2,2}(\mathbf{u}, \mathbf{x}_0, R/2) + \phi^{k+1,2}(p, \mathbf{x}_0, R/2) \leq$$
$$C\left(\frac{1}{R^2}\phi^{k+1}(\mathbf{u}, \mathbf{x}_0, R) + \phi^{k+1}(\mathbf{G}, \mathbf{x}_0, R) + \phi^{k+1}(h, \mathbf{x}_0, R)\right). \tag{4.18}$$

2. *The coefficients of* $\mathbf{A}(\mathbf{x})$ *and* $\mathbf{C}(\mathbf{x})$ *are in* $W^{1,p}(\Omega)$ *and* $\mathbf{A}$ *is uniformly symmetric positive definite and* $\mathbf{C}$ *satisfies the conditions of Lemma 4.3. Then suppose that* $(\mathbf{u}, p) \in C^1(\overline{\Omega}, \mathbb{E}^3) \times C(\overline{\Omega})$ *and* $\mathbf{G} \in H^1(\Omega, L(\mathbb{E}^3))$ *and* $h \in H^1(\Omega)$. *We also suppose that* $((\nabla \mathbf{u})\mathbf{j}, (\nabla p) \cot \mathbf{j}) \in H^1(K, \mathbb{E}^3) \times L^2(K)$ *for* $\mathbf{j} \cdot \mathbf{n} = 0$. *Then for compact* $K \subset \Omega \cup \Gamma$, $\mathbf{u} \in H^2(K, \mathbb{E}^3)$ *and* $p \in H^1(K)$ *and there exists some* $C(\delta, D) > 0$ *depending only on* $\delta > 0$, $D > \|\nabla \mathbf{u}\|_\infty + \|p\|_\infty$, *and the coefficients, such that for* $\mathbf{x}_0 \in \Omega \cup \Gamma$ *and* $R > 0$ *such that* $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) > R + \delta$ *then (4.18) holds with* $k = 0$.

3. *The coefficients of* $\mathbf{A}(\mathbf{x})$ *and* $\mathbf{C}(\mathbf{x})$ *are in* $W^{1,p}(\Omega)$ *and* $\mathbf{A}$ *is uniformly symmetric positive definite and* $\mathbf{C}$ *satisfies the conditions of Lemma 4.3. Then suppose that* $(\mathbf{u}, p) \in C^1(\overline{\Omega}, \mathbb{E}^3) \times C(\overline{\Omega})$ *and* $\mathbf{G} \in W^{1,p}(\Omega, L(\mathbb{E}^3))$ *and* $h \in W^{1,p}(\Omega)$. *We also suppose that* $((\nabla \mathbf{u})\mathbf{j}, (\nabla p) \cdot \mathbf{j}) \in W^{1,p}(K, \mathbb{E}^3) \times L^p(K)$ *for* $\mathbf{j} \cdot \mathbf{n} = 0$. *Then for compact* $K \subset \Omega \cup \Gamma$, $\mathbf{u} \in W^{2,p}(K, \mathbb{E}^3)$ *and* $p \in W^{1,p}(K)$.

Proof. Define the linear transformation $\mathbf{A}_n(\mathbf{x}) \in L(\mathbb{E}^3)$ by

$$\mathbf{b} \cdot \mathbf{A}_n(\mathbf{x})\mathbf{a} = \mathbf{b} \otimes \mathbf{n} \cdot \mathbf{A}(\mathbf{x})[\mathbf{a} \otimes \mathbf{n}]. \tag{4.19}$$

The transformation $\mathbf{A}_n(\mathbf{x})$ is positive definite because so is $\mathbf{A}(\mathbf{x})$ (this also follows just from the Legendre-Hadamard condition). The other notation we use is (and similar for scalars and tensors)

$$\begin{aligned}\nabla^{\perp}\mathbf{v} =&((\nabla\mathbf{v})\mathbf{n}) \otimes \mathbf{n}\\ \nabla^{\parallel}\mathbf{v} =&\nabla\mathbf{v} - \nabla^{\perp}\mathbf{v}.\end{aligned}$$

Next, we solve for the normal derivatives of p and $\nabla\mathbf{u}$. For case 1, suppose inductively we have shown that $(\mathbf{u}, p) \in H^{l+1}(K, \mathbb{E}^3) \times H^l(K)$ and $((\nabla\mathbf{u})\mathbf{j}, \mathbf{j} \cdot \nabla p) \in H^{l+1}(K, \mathbb{E}^3) \times H^l(K)$ for $\mathbf{j} \cdot \mathbf{n} = 0$ and $l \leq k$ and we will show they have one more derivative. Fix some collection of unit vectors $\{\mathbf{j}_i\}_{i=1}^{l}$ and let $\mathbf{w} = (\nabla^l\mathbf{u}((\mathbf{j}_1, \ldots, \mathbf{j}_l)$, $\mathbf{q} = \nabla^l p(\mathbf{j}_1, \ldots, \mathbf{j}_l)$, $\hat{\mathbf{G}} = \nabla^l\mathbf{G}(\mathbf{j}_1, \ldots, \mathbf{j}_l)$, $\hat{h} = \nabla h(\mathbf{j}_1, \ldots, \mathbf{j}_l)$ and solve first for $\mathbf{n} \cdot \nabla q$. In the first case remember that the coefficients are constant and in the second and third cases we only consider $l = 0$. We have a strong solution on the interior so compute pointwise:

$$\begin{aligned}\mathbf{C}(\mathbf{x})\nabla^{\perp}q =&\nabla \cdot \mathbf{A}(\mathbf{x})\nabla\mathbf{w} - \nabla \cdot \hat{\mathbf{G}} - \mathbf{C}(\mathbf{x})\nabla^{\parallel}q\\ =&\mathbf{A}_n\nabla^{\perp} \cdot \nabla^{\perp}\mathbf{w} + \mathbf{d}\end{aligned}$$

where $\mathbf{d}$ is a vector with terms either involving $\mathbf{A}$ product with second derivatives of $\mathbf{w}$ with at least one tangential component, $\nabla\mathbf{A}$ product with first derivatives of $\mathbf{w}$, first derivatives of $\mathbf{G}$ and $\mathbf{C}$ product with tangential derivatives of q. Next we can multiply by some invertible matrices

$$
\begin{aligned}
\mathbf{C}^\top \mathbf{A}_n^{-1} \mathbf{C} \nabla^\perp q =& \nabla^\perp (\mathbf{C}^\top \nabla^\perp \mathbf{w}) \mathbf{n} + \mathbf{d}_1 \\
(\mathbf{n} \cdot \mathbf{C}^\top \mathbf{A}_n^{-1} \mathbf{C} \mathbf{n})(\mathbf{n} \cdot \nabla q) =& \nabla^\perp (\mathbf{C} \cdot \nabla^\perp \mathbf{w}) \mathbf{n} + \mathbf{n} \cdot \mathbf{d}_1 \\
=& \mathbf{n} \cdot \nabla \hat{h} + d_2.
\end{aligned}
$$

and again d_2 is in terms of derivatives of already controlled terms. From that we can solve for $\nabla^\perp \cdot \nabla^\perp \mathbf{w}$ in the usual way by inverting $\mathbf{A}_n$. It is clear that $\mathbf{w}$ and $\mathbf{p}$ have at least one extra tangential derivative. Since this is done point-wise all the estimates remain for the normal derivatives. The inequality for the pressure follows immediately from the strong form of the equations. □

4.4.3 Constant Coefficient Estimates

With higher regularity, constant coefficient systems have smooth solutions. It is necessary to understanding the scaling of norms for these solutions to aid in further estimates.

Lemma 4.8 (Constant Coefficient Inequality)**.** *Let Ω and Γ be as in Lemma 4.6 and $(\mathbf{u}, p) \in H^1_\Gamma(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ be a solution to $S_{A,f}[\mathbf{u}, p] = (\iota[\mathbf{G}], h)$ with constant $\mathbf{A}$ and $\mathbf{C}$, and constant $\mathbf{G}$ and h. Then with $0 < \rho < R$ and $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) > R + \delta$, the three following inequalities are satisfied with C depending only on k, δ and the coefficients:*

1. *For integer $k \geq 0$*

$$
\phi^{k,2}(p, \mathbf{x}_0, \rho) + \phi^{k+1,2}(\mathbf{u}, \mathbf{x}_0, \rho) \leq C(k, \delta) \left(\frac{\rho}{R} \right)^n \phi^{k+1,2}(\mathbf{u}, \mathbf{x}_0, R) \tag{4.20}
$$

2.

$$\psi^{0,2}(p, \mathbf{x}_0, \rho) + \psi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho) \leq C(\delta) \left(\frac{\rho}{R}\right)^{n+2} \psi^{1,2}(\mathbf{u}, \mathbf{x}_0, R). \tag{4.21}$$

Proof. This lemma is a pretty straight forward corollary of the Caccioppoli inequality, higher regularity, and Poincaré inequaliy with some neat scaling tricks and a trick to add the oscillation.

First we prove (4.20). Higher regularity with constant coefficients, Sobolev embedding, and Caccioppoli imply that for some $d > 0$, depending on k and the dimension $n = 3$,

$$\begin{aligned}
&\sup_{\mathbf{x} \in B(\mathbf{x}_0, 2^{-d}R, \Omega)} |\nabla^{k-1} p(\mathbf{x})| + \sup_{\mathbf{x} \in B(\mathbf{x}_0, 2^{-d}R, \Omega)} |\nabla^k \mathbf{u}(\mathbf{x})| \\
\leq & C(R, k) \left(\phi^{d+k,2}(\mathbf{u}, \mathbf{x}_0, 2^{-2}R) + \phi^{d+k-1,2}(p, \mathbf{x}_0, 2^{-2}R)\right) \\
\leq & C(R, k) \phi^{k,2}(\mathbf{u}, \mathbf{x}_0, R).
\end{aligned} \tag{4.22}$$

from which we see that for $\rho \leq 2^{-d}R$, and by extension $\rho \leq R$, (the constant still depends on k but this is not made explicit)

$$\phi^{k,2}(\mathbf{u}, \mathbf{x}_0, \rho) \leq C(R)\rho^n \phi^{k,2}(\mathbf{u}, \mathbf{x}_0, R).$$

Next rescale by $\tilde{\mathbf{x}} = (\mathbf{x} - \mathbf{x}_1)/R$ so that $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \mathbf{u}(R\tilde{\mathbf{x}})$ also solves a constant coefficient system in $\tilde{\Omega} = \{(\mathbf{x} - \mathbf{x}_1)/R : \mathbf{x} \in \Omega\}$. We change variables and apply (4.22) with ρ/R, then change variables back to prove (4.20):

$$\begin{aligned}\phi^{k,2}(\mathbf{u},\mathbf{x}_0,\rho) =& R^n \int_{B(\mathbf{0},\rho/R,\tilde{\Omega})} R^{-2k}|\tilde{\nabla}^k\tilde{\mathbf{u}}|^2 dV\\ \leq& R^n C(1)\left(\frac{\rho}{R}\right)^n \int_{B(\mathbf{0},1,\tilde{\Omega})} R^{-2k}|\tilde{\nabla}\tilde{\mathbf{u}}|^2 dV_{\mathbf{y}}\\ \leq& C(1)\left(\frac{\rho}{R}\right)^n \phi^{k,2}(\mathbf{u},\mathbf{x}_0,R).\end{aligned}$$

Now we prove (4.21). Define $\mathbf{U}(\mathbf{x}) = (\nabla\mathbf{u})_{B(\mathbf{x}_0,R,\Omega)}\mathbf{x}$ so that $\nabla\mathbf{U}(\mathbf{x}) = (\nabla\mathbf{u})_{B_R}B(\mathbf{x}_0,R,\Omega)$ and $P(\mathbf{x}) = p(\mathbf{x}) - (P)_{B_R}$ and then $(\mathbf{u}-\mathbf{U}, p-P)$ solves

$$S_{\mathbf{A},f}[\mathbf{u}-\mathbf{U}, p-P] = (\mathbf{G} - \iota[\mathbf{A}(\nabla\mathbf{u})_{B(\mathbf{x}_0,R,\Omega)}], h - \mathbf{C}(\nabla\mathbf{u})_{B(\mathbf{x}_0,R,\Omega)}).$$

From the Lemma 4.6, the estimate of higher derivatives does not depend on the constant terms added.

First we use the Poincaré inequality for $\rho < R/2$,

$$\psi^{0,2}(p,\mathbf{x}_0,\rho) + \psi^{1,2}(\mathbf{u},\mathbf{x}_0,\rho) \leq C\rho^2\left(\phi^{1,2}(p,\mathbf{x}_0,\rho) + \phi^{2,2}(\mathbf{u},\mathbf{x}_0,\rho)\right)$$

then from (4.20),

$$C\rho^2\left(\phi^{1,2}(p,\mathbf{x}_0,\rho) + \phi^{2,2}(\mathbf{u},\mathbf{x}_0,\rho)\right) \leq C\rho^2\left(\frac{\rho}{R}\right)^n \phi^{2,2}(\mathbf{u}-\mathbf{U},\mathbf{x}_0,R/2)$$

and higher regularity, (4.18), for $\mathbf{u}-\mathbf{U}$ implies that

$$C\rho^2\left(\frac{\rho}{R}\right)^n \phi^{2,2}(\mathbf{u}-\mathbf{U},\mathbf{x}_0,R/2) \leq C\left(\frac{\rho}{R}\right)^{n+2} \phi^{1,2}(\mathbf{u}-\mathbf{U},\mathbf{x}_0,R), \tag{4.23}$$

which completes the argument because $\phi^{1,2}(\mathbf{u}-\mathbf{U},\mathbf{x}_0,R) = \psi^{1,2}(\mathbf{u},\mathbf{x}_0,R)$. □

4.4.4 Variable Coefficient Estimates

From the perspective of the Hölder space estimates, variable coefficients may be dealt with by 'freezing the coefficients' at a point and using control over modulus of continuity along with constant coefficient estimates and localization lemmas to control the norms. At the boundary it is still necessary to map to a domain with flat boundary as we have done for the last two lemmas.

Lemma 4.9 (Variable Coefficient Inequalities)**.** *Let* $(\mathbf{u}, p)$ *be a solution to (4.9) in* Ω *and* $\omega^2(R) = \sup_{\mathbf{x}\in\overline{\Omega}} \omega^2(|\mathbf{A}|, \mathbf{x}, R) + \sup_{\mathbf{x}\in\overline{\Omega}} \omega^2(|\mathbf{C}|, \mathbf{x}, R)$*. Suppose that* $\Gamma \subset \Omega$ *is flat as in Lemma 4.6. Define*

$$D(\mathbf{x}_0, R) = \psi^{0,2}(\mathbf{G}, \mathbf{x}_0, R) + \psi^{0,2}(h, \mathbf{x}_0, R). \tag{4.24}$$

Then there is C *and* R_0 *such that for* $0 \leq \rho \leq R \leq R_0$ *and for* $\mathbf{x}_0 \in \overline{\Omega}$ *with* $d(\mathbf{x}_0, \partial\Omega\backslash\Gamma) \geq R + \delta$

$$\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho) \leq C(\delta)\left(\left[\left(\frac{\rho}{R}\right)^n + \omega^2(R)\right]\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + D(\mathbf{x}_0, R)\right) \tag{4.25}$$

and

$$\begin{aligned}&\psi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho) + \psi^{0,2}(p, \mathbf{x}_0, \rho)\\ \leq &C(\delta)\left(\left(\frac{\rho}{R}\right)^{n+2}\psi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + \omega^2(R)\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + D(\mathbf{x}_0, R)\right)\end{aligned} \tag{4.26}$$

Proof. Pick R and $\mathbf{x}_0$ so that $R+\delta < d(\mathbf{x}_0, \partial\Omega\backslash\Gamma)$. Find $(\mathbf{w}_1, q_1)$ that solve the homogenized problem with coefficients frozen at $\mathbf{x}_0$. That is $\mathbf{w}_0 = \mathbf{u}-\mathbf{w}_1 \in H^1_0(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$

and

$$\int_{B(\mathbf{x}_0,R,\Omega)} (\nabla \mathbf{v}) \cdot [\mathbf{A}(\mathbf{x}_0)\nabla \mathbf{w}_1 - q_1 \mathbf{C}(\mathbf{x}_0) - (\mathbf{G})_{B(R,\mathbf{x}_0,\Omega)}]\, dV = 0, \ \forall \mathbf{v} \in H_0^1(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$$

$$\mathbf{C}(\mathbf{x}_0) \cdot \nabla \mathbf{w}_1 = (g)_{B(\mathbf{x}_0,R,\Omega)}. \tag{4.27}$$

Since $\mathbf{u}$ vanishes on Γ, so does $\mathbf{w}_1$, and applying the inequalities from Lemma 4.8 we have estimates for the L^2 and L^2-oscillation semi-norms of $\nabla \mathbf{w}_1$. For any $\rho < R$,

$$\begin{aligned} \phi^{1,2}(\mathbf{w}_1, \mathbf{x}_0, \rho) &\le C \left(\frac{\rho}{R}\right)^n \phi^{1,2}(\mathbf{w}_1, \mathbf{x}_0, R), \\ \psi^{1,2}(\mathbf{w}_1, \mathbf{x}_0, \rho) + \psi^{0,2}(q_1, \mathbf{x}_0, \rho) &\le C \left(\frac{\rho}{R}\right)^{n+2} \psi^{1,2}(\mathbf{w}_1, \mathbf{x}_0, R). \end{aligned} \tag{4.28}$$

For $\mathbf{w}_0$, $\mathbf{C}(\mathbf{x}_0) \cdot \nabla \mathbf{w}_0 = g - (g)_R - [\mathbf{C}(\mathbf{x}) - \mathbf{C}(\mathbf{x}_0)] \cdot \nabla \mathbf{u}$ and for $q_0 = p - q_1$ and $\mathbf{v} \in H_0^1(B(\mathbf{x}_0, R, \Omega), \mathbb{E}^3)$

$$\begin{aligned} &\int_{B(\mathbf{x}_0,R,\Omega)} (\nabla \mathbf{v}) \cdot [\mathbf{A}(\mathbf{x}_0)\nabla \mathbf{w}_0 - q_0 \mathbf{C}(\mathbf{x}_0)]\, dV \\ =&\int_{B(\mathbf{x}_0,R,\Omega)} (\nabla \mathbf{v}) \cdot [\mathbf{G} - (\mathbf{G})_R - (\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x_0}))\, \nabla \mathbf{u}]\, dV. \end{aligned} \tag{4.29}$$

Then use coercivity and pressure control, Lemma 4.4. Plug in $\mathbf{w}_0$ as the test function to get the estimate

$$\phi^{1,2}(\mathbf{w}_0, \mathbf{x}_0, R) \le C \left(\omega^2(R) \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + \psi^{0,2}(\mathbf{G}, \mathbf{x}_0, R) + \psi^{0,2}(h, \mathbf{x}_0, R) \right). \tag{4.30}$$

Estimate (4.25) now follows almost immediately from the triangle inequality for $\mathbf{u}$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_0$.

For (4.26), estimate the L^2 oscillation of $\nabla\mathbf{u}$ and p,

$$\begin{aligned}
&\psi^{1,2}(\mathbf{u},\mathbf{x}_0,\rho)+\psi^{0,2}(p,\mathbf{x}_0,\rho)\\
\le&2\psi^{1,2}(\mathbf{w}_1,\mathbf{x}_0,\rho)+2\psi^{0,2}(q_1,\mathbf{x}_0,\rho)\\
&+2\psi^{1,2}(\mathbf{w}_1,\mathbf{x}_0,\rho)+2\psi^{0,2}(q_0,\mathbf{x}_0,\rho)\\
\le&C\left(\left(\frac{p}{R}\right)^{n+2}\psi^{1,2}(\mathbf{w}_1,\mathbf{x}_0,R)+\phi^{1,2}(\mathbf{w}_0,\mathbf{x}_0,R)\right)\\
\le&C\left(\left(\frac{p}{R}\right)^{n+2}\psi^{1,2}(\mathbf{u},\mathbf{x}_0,R)+\phi^{1,2}(\mathbf{u},\mathbf{x}_0,R)+D(\mathbf{x}_0,R)\right)
\end{aligned}$$

showing (4.26). □

4.4.5 Morrey and Campanato Spaces

The Campanato spaces, with oscillatory integral norms, are shown to be equivalent in some cases to the Hölder spaces. This provides an important connection between the modulus of continuity and the scaling of various integral norms.

Definition 4.3. *Let $\rho_1 = \mathrm{diam}(\Omega)$. The Morrey space $L^{p,\lambda}(\Omega)$ for $1 \le p < \infty$ is the set of elements $w \in L^p(\Omega)$ such that $\|w\|_{L^{p,\lambda}(\Omega)} < \infty$ where*

$$\|w\|^p_{L^{p,\lambda}(\Omega)} = \sup_{\mathbf{x}_0\in\Omega,\ 0<\rho<\rho_1} \rho^{-\lambda}\phi^{0,p}(u,\mathbf{x}_0,\rho). \tag{4.31}$$

This is a norm such that $L^{p,\lambda}(\Omega)$ is a Banach space. A few properties are, assuming that Ω satisfies the cone condition:

- *(generalized Hölder inequality) $L^{q,\mu}(\Omega) \subset L^{p,\lambda}(\Omega)$ if $\frac{n-\lambda}{p} \ge \frac{n-\mu}{q}$ and $p \le q$.*

- *$L^{p,0}(\Omega) = L^p(\Omega)$ and $L^{p,n}(\Omega) = L^\infty(\Omega)$ for any p.*
- *If $\lambda > n$ then $L^{p,\lambda}(\Omega) = \{0\}$.*

The Campanato semi-norm is

$$[w]^p_{p,\lambda} = \sup_{\mathbf{x}_0 \in \Omega,\ 0<\rho<\rho_1} \rho^{-\lambda} \psi^{0,p}(w, \rho, \mathbf{x}_0). \tag{4.32}$$

We define the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ to be the subspace of $L^p(\Omega)$ with norm

$$\|w\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|w\|_{L^p(\Omega)} + [w]_{p,\lambda}. \tag{4.33}$$

Again this is a Banach space with the following properties:

- *$\mathcal{L}^{q,\mu}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$ if $\frac{n-\lambda}{p} \geq \frac{n-\mu}{q}$ and $p \leq q$.*
- *$L^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$ and equality holds for $0 \leq \lambda < n$.*
- *For all $p \in [1, \infty)$, $\mathcal{L}^{p,n}(\Omega) = BMO(\Omega)$ and for $n < \lambda \leq n + p$ let $\alpha = \frac{\lambda-n}{p}$ and $\mathcal{L}^{p,\lambda}(\Omega) = C^{0,\alpha}(\overline{\Omega})$. For $\lambda > n + p$ the Campanato space consists only of constant functions.*

Lemma 4.10. *Suppose $(\mathbf{u}, p) \in H^1_\Gamma(\Omega, \mathbb{E}^3) \times L^2(\Omega)$ are a solution to (4.9) in Ω with $\Gamma \subset \partial\Omega$ flat and $\mathbf{A}$, $\mathbf{C}$ continuous and satisfying the coercivity and surjectivity properties. Let K be a compact subset of $\Omega \cup \Gamma$ with $\operatorname{dist}(K, \partial\Omega \backslash \Gamma) > 0$.*

1. *If $\mathbf{G} \in L^{2,\mu}(\Omega, L(\mathbb{E}^n))$ and $h \in L^{2,\mu}(\Omega)$ for $0 \leq \mu < n$ then $\nabla \mathbf{u} \in L^{2,\mu}(K, L(\mathbb{E}^n))$.*
2. *If instead $\mathbf{A}, \mathbf{C}$ Hölder continuous with exponent α on $\overline{\Omega}$, and $\mathbf{G} \in \mathcal{L}^{2,\mu}(\Omega, L(\mathbb{E}^n))$ and $h \in \mathcal{L}^{2,\mu}(\Omega)$ for $n \leq \mu < n + 2$ and $\alpha = \frac{1}{2}(\mu - n)$, then $\nabla \mathbf{u} \in \mathcal{L}^{2,\mu}(K, L(\mathbb{E}^n))$ and $p \in \mathcal{L}^{2,\mu}(K)$.*

Proof. For 1., fix $\mu < n$. Let $B = \|\mathbf{G}\|^2_{\mathcal{L}^{2,\mu}(\Omega)} + \|h\|^2_{\mathcal{L}^{2,\mu}(\Omega)} \geq D(\mathbf{x}_0, R)R^{-\mu}$ by definition of the Campanato space and equivalence of norm with the Morrey space for $\mu < n$. For $\mathbf{x}_0 \in \Omega \cup \Gamma$ with $d(\mathbf{x}_0, \partial\Omega \backslash \Gamma) \geq R + \delta$ then inequality from the Lemma 4.9 has the form

$$\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho) \leq C \left[\left(\frac{\rho}{R}\right)^n + \omega^2(R) \right] \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + BR^{\mu}, \; \forall 0 < \rho \leq R \leq R_0. \tag{4.34}$$

By iteration, Lemma 4.13, we conclude there is an $\epsilon_0 > 0$ such that whenever $\omega^2(R) < \epsilon_0$,

$$\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho)\rho^{-\mu} \leq C_1 \left[R^{-\mu} \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + B \right]. \tag{4.35}$$

For $\omega^2(R) > \epsilon_0$, fix R_1 with $\omega^2(R_1) < \epsilon_0$. If $\rho \leq R_1 \leq R \leq R_0$, then

$$\begin{aligned} \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho)\rho^{-\mu} \leq & C_1[R_1^{-\mu}\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R_1) + B] \\ \leq & C_2[R^{-\mu}\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) + B] \end{aligned}$$

where C_2 incorporates $R_0^{\mu}R_1^{-\mu}$. For $R_1 \leq \rho \leq R \leq R_0$ we have immediately

$$\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho)\rho^{-\mu} \leq R_1^{-\mu}\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R)$$

so inequality (4.35) holds for $\rho \leq R < R_0$ and it follows that $\nabla \mathbf{u} \in L^{2,\mu}(K, L(\mathbb{E}^n))$.

For part 2., we have $\nabla \mathbf{u} \in L^{2,\mu_0}(\Omega, L(\mathbb{E}^n))$ for $\mu_0 = n - \epsilon$ and any $\epsilon > 0$ by part 1. Then we abbreviate

$$\psi(\mathbf{x}_0, \rho) = \psi^{1,2}(\mathbf{u}, \mathbf{x}_0, \rho) + \psi^{0,2}(p, \mathbf{x}_0, \rho), \tag{4.36}$$

and let

$$\begin{aligned} B(\mu_0, \mu_1) = & \|\mathbf{G}\|^2_{\mathcal{L}^{2,\mu_1}(\Omega, L(\mathbb{E}^3))} + \|h\|^2_{\mathcal{L}^{2,\mu_1}(\Omega)} + \sup_{0<R<R_1} \{R^{-2\alpha}\omega^2(R)\} \|\nabla \mathbf{u}\|^2_{L^{2,\mu_0}(K, L(\mathbb{E}^3))} \\ \geq & R^{-\mu_1} \left(D(\mathbf{x}_0, R) + R^{-2\alpha - \mu_0 + \mu_1} \omega^2(R) \phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) \right). \end{aligned}$$

There are two cases. If $n-2\alpha < \mu_0 < n$ we choose $n \le \mu_1 \le \mu_0+2\alpha$ and the oscillation inequality (4.26) gives

$$\psi(\mathbf{x}_0,\rho) \le C\left[\left(\frac{\rho}{R}\right)^{n+2}\psi(\mathbf{x}_0,R) + B(\mu_0,\mu_1)R^{\mu_1}\right],\ \forall 0<\rho\le R\le R_0 \tag{4.37}$$

and apply the iteration lemma to find a constant for which

$$\psi(\mathbf{x}_0,\rho)\rho^{-\mu_1} \le C\left[R^{-\mu_1}\psi(\mathbf{x}_0,R) + B(\mu_0,\mu_1)\right]. \tag{4.38}$$

This shows that $\nabla\mathbf{u} \in \mathcal{L}^{2,\mu_1}(\Omega, L(\mathbb{E}^3))$ and the same for p. The case $\mu_1 = n$ is important for interpolation and $\mu > n$ are Hölder spaces. This is not quite the best estimate because we only reach $\mu_1 = n + 2\alpha - \epsilon$ for some $\epsilon > 0$.

With an extra step we get optimal regularity for the largest $\mu > n$ such that $\omega^2(R)R^{-\mu+n}$ is bounded ($\mu = n + 2\alpha$) and the data is in $\mathcal{L}^{2,\mu}(\Omega)$. The previous step implies that $\nabla\mathbf{u} \in C^0(\overline{\Omega}, L(\mathbb{E}^3))$ so that $\left(R^{n-\mu}\omega^2(R)\right)\left(R^{-n}\phi^{1,2}(\mathbf{u},\mathbf{x}_0,R)\right)$ is bounded for small R. Finally, apply the iteration inequality for $B(n,\mu)$ so that

$$\psi(\mathbf{x}_0,\rho) \le C\left[\left(\frac{\rho}{R}\right)^{n+2}\psi(\mathbf{x}_0,R) + B(n,\mu)R^{\mu}\right] \tag{4.39}$$

implies

$$\psi(\mathbf{x}_0,\rho)\rho^{-\mu} \le C\left[R^{-\mu}\psi(\mathbf{x}_0,R) + B(n,\mu)\right],\ \forall\ 0<\rho\le R\le R_0 \tag{4.40}$$

and $B(n,\mu)$ is bounded by that $\mathcal{L}^{2,\mu}$ norms of the data and the α Hölder norm of the coefficients, as well as the supremum of $|\nabla\mathbf{u}|$. □

4.4.6 L^p Interpolation

Here is a statement of the interpolation theorem from [18].

Theorem 4.2 (Stampacchia Interpolation Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and Q a subset of $\mathbb{R}^n$. Suppose $T : L^p(\Omega) \to L^p(Q)$ is bounded for $1 \le p < \infty$, and T is also bounded from $T : L^\infty(\Omega) \to \mathcal{L}^{p,n}(Q) = BMO(Q)$. Then for all $p \le q < \infty$, T is bounded from $L^q(\Omega) \to L^q(Q)$, with norm dependent on n, p, q then bounds on the the two spaces and $|\Omega|/|Q|$.*

We make use of this theorem in the proof of Theorem 4.1 for the solution operator to the Stokes-equation mapping to the gradient of $\mathbf{u}$. For $p = 2$ we have shown global (and local) well-posedness in Proposition 4.1. Then Lemma 4.10 shows continuity for the solution operator into the BMO space making use of the continuous embedding $L^\infty(\Omega) \to BMO(\Omega)$.

4.5 Other Technical Results

We state these and the proofs can be found in for example [17] or [9] as well as many other sources that deal with regularity theory.

Lemma 4.11 (Poincaré Inequality)**.** *Suppose $\mathbf{u} \in H^1_0(\Omega, \mathbb{E}^n)$, then*

$$\frac{1}{R^2}\psi^{1,2}(\mathbf{u}, \mathbf{x}_0, R) \le C\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R), \tag{4.41}$$

or if $\mathbf{x}_0 \in \partial\Omega$*, then for a constant depending on* Ω*,*

$$\frac{1}{R^2}\phi^{0,2}(\mathbf{u}, \mathbf{x}_0, R) \leq C(\Omega)\phi^{1,2}(\mathbf{u}, \mathbf{x}_0, R). \tag{4.42}$$

Lemma 4.12 (ϵ-lemma)**.** *Let* f, g, h *be non-negative functions in* $L^1(B(\mathbf{0}, 1))$ *and* $\alpha > 0$*. Suppose there exits* ϵ_0 *such that for all* $\epsilon < \epsilon_0$ *there is an inequality (for balls centered at* $\mathbf{x}_0$ *and* $R < \operatorname{dist}(\mathbf{x}_0, \partial B(\mathbf{0}, 1))$*) :*

$$\int_{B(\mathbf{x}_0,R)} f\, dV \leq C(\epsilon)\left(R^{-\alpha}\int_{B(\mathbf{x}_0,2R)} g\, dV + \int_{B(\mathbf{x}_0,2R)} h\, dV\right) + \epsilon\int_{B(\mathbf{x}_0,2R)} f\, dV \tag{4.43}$$

then there is a constant C depending only on α *and n (dimension) such that*

$$\int_{B(\mathbf{x}_0,R)} f\, dV \leq C\left(R^{-\alpha}\int_{B(\mathbf{x}_0,2R)} g\, dV + \int_{B(\mathbf{x}_0,2R)} h\, dV\right). \tag{4.44}$$

Lemma 4.13 (Iteration Lemma)**.** $\Phi(\rho)$ *(i.e.* $\phi(\mathbf{u}, \mathbf{x}_0, \rho)$*) is nonnegative and nondecreasing. Suppose that for constants A (arbitrary constant),* α *(n or* $n + 2$*),* β *(*μ*),* R_0*, B (norm of data),* ϵ *(the modulus of continuity of the coefficients) with* $\beta < \alpha$

$$\Phi(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \epsilon\right]\Phi(R) + BR^{\beta},\ \forall\ 0 < \rho \leq R \leq R_0 \tag{4.45}$$

then there existst an ϵ_0 *and* $C(A, \alpha, \beta)$ *such that if* $\epsilon < \epsilon_0$ *then*

$$\Phi(\rho) \leq C\left[\left(\frac{\rho}{R}\right)^{\beta}\Phi(R) + B\rho^{\beta}\right],\ \forall\ 0 < \rho \leq R \leq R_0. \tag{4.46}$$

4.6 Conclusion

We focus on the regularity theory needed in Chapter 2, however there is much much more theory that is relevant to our problem, as well as much more work to be done on the subject. In particular, we are interested in the higher regularity for weak solutions to the linearized equations that we consider in Chapter 2. Higher regularity for weak elliptic systems has been studied, for example [16], however there is some work to do to adapt the methods to the incompressibility constraints. There are also many results on partial regularity for non-linear systems and it would be interesting to carry out an analysis of how partial regularity behaves with the presence of an incompressibility constraint. This is a very difficult subject, however, and we do not claim the expertise required to make significant contributions. In [12] it is recognized that quasi-convexity is an essential ingredient for partial regularity. For the non-linear incompressibility constraint we expect to need a related property, for example a non-linear version of Lemma 4.3.

Much of the regularity theory for non-linear elliptic systems stagnates at a point where the maximum principle performs a lot of work in the case of scalar equations. Famously, a maximum principle is applied for non-linear systems relating to Ricci flow in [22], which takes advantage of significantly more geometric structure. A hopeful sign that the maximum principle could be used in the context of non-linear elasticity came in [6]. Despite much effort by those authors and others, this maximum principle was not utilized in significant generality and it is very difficult to anticipate what will bring the next breakthrough.

APPENDIX A

FRÉCHET DIFFERENTIABILITY OF NEMYTSKII OPERATORS

We collect some results on continuity, measurability, and differentiability of Nemytskii operators that are needed for the non-linear analysis of Chapters 2 and 3. There are three cases to show continuous Fréchet differentiability we need. We assume that Ω is an open and bounded subset of $\mathbb{E}^n$.

1. We consider maps from $L^q(\Omega)$ to $L^r(\Omega)$ with $1 \leq r < q$. This case is used to show the stored energy (2.2) is continuously Fréchet differentiable from $W^{2,p}(\Omega, \mathbb{E}^3) \to \mathbb{R}$. The failure of continuity when $r = q$ is also remarkable.
2. Next we show when a map from $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is continuously differentiable for $p > n$. This is used to show the determinant operator (2.3) is continuously differentiable. This case also shows up frequently when changing variables to show compositions operators are continuous. Continuity fails when $p \leq n$ with the failure of the embedding into a Hölder space.
3. Last we show when maps from $C^{0,\alpha}(\Omega) \to C^{0,\alpha}(\Omega)$ are continuously differentiable. We allow any $\alpha \in [0, 1)$ although the special case $\alpha = 0$ is considerably simpler.

A.1 Definitions

Let $A : E \to F$ be a map between Banach spaces.

Definition A.1. *The map A is* Fréchet differentiable *at a point $e \in E$ if there exits a continuous linear operator $T_e \in BL(E, F)$ such that*

$$\limsup_{h \to 0, h \neq 0} \frac{\|A[e+h] - A[e] - T_e h\|_F}{\|h\|_E} = 0. \tag{A.1}$$

Definition A.2. *We say A is* continuously (Fréchet) differentiable *(or C^1) at e_0 if there is an open (norm topology) neighborhood O of e_0 such that for all $e_1 \in O$, A is Fréchet differentiable with derivative $DA[e]$, and the derivative map from $O \to BL(E, F)$ is continuous with respect to the operator norm for the co-domain and the norm for E. A quantified statement is that for all $\epsilon > 0$ and $e_1 \in O$ there exists $\delta > 0$ such that for all $h \in E$ and $\|e_1 - e_0\|_E < \delta$,*

$$\|DA[e_1]h - DA[e_0]h\|_F \leq \epsilon \|h\|_E. \tag{A.2}$$

Remark A.1. *Definitions A.1 and A.2 immediately imply that a continuous linear operator is C^1 and equal to its derivative.*

Definition A.3. *We call A* Gâteaux differentiable *at $e \in E$, if there is a continuous linear operator $T_e \in BL(E, F)$ such that for all $h \in E$*

$$\lim_{\lambda \to 0} \frac{\|A[e + \lambda h] - A[e] - \lambda T_e h\|_F}{\lambda} = 0. \tag{A.3}$$

Remark A.2. *In some of the literature the Gâteaux derivative is not assumed to be linear in h. In [30], this additional assumption is expressed as "linear in its increments". For this thesis, the more important distinction will be the subtle one between Definitions A.1 and A.3. The distinction is that the limit need not be uniform for $h \in E$ for a map to have a Gâteaux derivative. This is especially important when the norm topology on E is either not relevant or too strong as the E norm shows up in (A.1) but not (A.3).*

Next we introduce the relevant setting for Nemytskii operators. Suppose E and F are Banach spaces of functions on Ω with co-domain $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. Let $\mathbf{f} : \mathbb{R}^N \times \Omega \to \mathbb{R}^M$.

Definition A.4. *The function* $\mathbf{f}$ *is* Carathéodory *if* $\mathbf{w} \mapsto \mathbf{f}$ *is continuous and* $\mathbf{x} \mapsto \mathbf{f}$ *is measurable. Furthermore, the modulus of continuity is uniform on* Ω *in the sense that there is a function* $\omega : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *such that* $\lim_{h\to 0} \omega(\mathbf{e}_0, h) = 0$ *and* $|\mathbf{f}(\mathbf{e}_0, \mathbf{x}) - \mathbf{f}(\mathbf{e}_1, \mathbf{x})| < \omega(\mathbf{e}_0, |\mathbf{e}_0 - \mathbf{e}_1|)$ *for all* $\mathbf{x} \in \Omega$ *and* $\mathbf{e}_0, \mathbf{e}_1 \in \mathbb{R}^N$*.*

Remark A.3. *Without the uniformity condition the map* $\mathbf{f}$ *is still jointly measurable and some weak continuity will still be satisfied. We choose to include the uniformity in the definition because it is required for continuity with respect to the norm topologies.*

Definition A.5. *We define* $N : E \to F$ *to be a* Nemystkii operator *if it has the form* $N[\mathbf{e}](\mathbf{x}) = \mathbf{f}(\mathbf{e}(\mathbf{x}), \mathbf{x})$ *for* $\mathbf{x} \in \Omega$*, where* $\mathbf{f} : \mathbb{R}^N \times \Omega \to \mathbb{R}^M$ *is Carathéodory.*

A.2 Simple Failure Conditions

An interesting and useful example when a Nemystkii operator fails to be continuous happens with $E = L^p(\Omega)$, $F = L^q(\Omega)$, and $p \leq q \leq \infty$. Then N is continuously differentiable if and only if $\mathbf{f}$ is linear with respect to $\mathbb{R}^N$. A proof of this can be found in [11] and in [10] necessary conditions (only if) are proven for the Hölder space setting. We now prove the necessary condition for a simple setting and then focus on sufficient conditions of $\mathbf{f}$ for N to be continuously differentiable.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly C^1, then for $e \in L^p(\Omega)$, the composition $N[e](\mathbf{x}) = f(e(\mathbf{x}))$ satisfies $N[e] \in L^\infty(\Omega)$ because f is bounded and continuity of f implies $N[e]^{-1}((a,\infty)) = e^{-1}(f^{-1}((a,\infty)))$ is measurable. This implies that N is everywhere Fréchet differentiable as a map from $L^p(\Omega) \to L^p(\Omega)$ with derivative $DN[e]h(\mathbf{x}) = f'(e(\mathbf{x}))h(\mathbf{x})$ for $h \in L^p(\Omega)$ because $\|DN[e]h\|_{L^p(\Omega)} \leq \|f'\|_{L^\infty(\mathbb{R})}\|h\|_{L^p(\Omega)}$ and

$$\|N[e] - N[e+h] - DN[e]h\|_{L^p(\Omega)} \leq \int_\Omega \omega(h(\mathbf{x}))dV. \tag{A.4}$$

However, the derivative is generally not continuous as the map $N' : L^p(\Omega) \to L^\infty(\Omega)$ by $N'[e](\mathbf{x}) = f'(e(\mathbf{x}))$ is not continuous unless f' is constant. This is the simplest case of failure of continuity and here is a short proof:

Lemma A.1. *Suppose $f \in C^0(\mathbb{R})$. Then the composition map $N : L^p(\Omega) \to L^\infty(\Omega)$ by $N[\phi](\mathbf{x}) = f(\phi(\mathbf{x}))$ for $1 \leq p < \infty$ is continuous if and only if f is constant.*

Proof. Suppose f is non-constant so that there is $\alpha, \beta \in \mathbb{R}$ with $f(\alpha) = a$ and $f(\beta) = b$ with $a \neq b$. For some point $\mathbf{x}_0 \in \Omega$ and $\epsilon > 0$ such that $B(\mathbf{x}_0, \epsilon) \subset \Omega$, choose

$$\phi(\mathbf{x}) = \left\{ \begin{array}{c|l} \alpha & |\mathbf{x} - \mathbf{x}_0| \leq \epsilon \\ 0 & |\mathbf{x} - \mathbf{x}_0| > \epsilon \end{array} \right. \quad \text{and} \quad \psi(\mathbf{x}) = \left\{ \begin{array}{c|l} \beta & |\mathbf{x} - \mathbf{x}_0| \leq \epsilon \\ 0 & |\mathbf{x} - \mathbf{x}_0| > \epsilon \end{array} \right. .$$

Then $\|\phi - \psi\|_{L^p(\Omega)} = \omega_n \epsilon^n |\alpha - \beta|$ and $\|f(\phi) - f(\psi)\|_{L^\infty} = |a - b|$. As $\epsilon \to 0$ the distance of the functions in the domain becomes arbitrarily small but the distance in $L^\infty(\Omega)$ remains constant. This shows the map is not continuous and it is continuous only if f is constant. $\square$

A.3 Continuity and Fréchet Differentiability Results

In order to prove Fréchet differentiability of composition operators it helps to first have some continuity results.

A.3.1 Norm of Product Operators

Lemma A.2 encompasses some well know results about products of linear operators.

Lemma A.2. *1. Suppose $f \in L^p(\Omega)$ with $1 \le p < \infty$ and $1 \le q < \infty$ such that r defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ satisfies $1 \le r < \infty$. Let $E = L^q(\Omega)$ and $F = L^r(\Omega)$, then the (linear) Nemystkii operator given by $N[e](\mathbf{x}) = f(\mathbf{x})e(\mathbf{x})$ is continuous (norm topologies) with norm $\|N\|_{BL(E,F)} \le \|f\|_{L^p(\Omega)}$.*

2. Suppose $f \in W^{1,p}(\Omega)$ for $p > n$, then the product map, a Nemystkii operator with $E = F = W^{1,p}(\Omega)$, $N[e](\mathbf{x}) = f(\mathbf{x})e(\mathbf{x})$ is continuous with operator norm bounded by $\|f\|_{W^{1,p}(\Omega)}$.

3. Suppose $f \in C^{0,\alpha}(\Omega)$ for $0 < \alpha < 1$, then for $E = F = C^{0,\alpha}(\Omega)$ and $N[e](\mathbf{x}) = f(\mathbf{x})e(\mathbf{x})$, N is continuous with operator norm bounded by the $C^{0,\alpha}$ norm of f.

Proof. 1. By Hölder's inequality

$$\|N[e]\|^r_{L^r(\Omega)} = \|(fe)^r\|_{L^1(\Omega)} \le \|f^r\|_{L^{p/r}(\Omega)}\|e^r\|_{L^{q/r}(\Omega)} = \|f\|^r_{L^p(\Omega)}\|e\|^r_{L^q(\Omega)}$$

and the linear operator is continuous because it is bounded.

2. It suffices to only consider the term of the norm with highest derivatives as the same calculation would work with room to spare for lower derivatives:

$$
\begin{aligned}
\|(\nabla f(\mathbf{x}))e(\mathbf{x}) + f(\mathbf{x})\nabla e(\mathbf{x})\|_{L^p(\Omega)} \leq & \|f\|_{W^{1,p}(\Omega)}\|e\|_{C^0(\Omega)} + \|e\|_{W^{1,p}(\Omega)}\|f\|_{C^0(\Omega)} \\
\leq & 2\|f\|_{W^{1,p}(\Omega)}\|e\|_{W^{1,p}(\Omega)}.
\end{aligned}
$$

3. Denote by $[f]_{C^{0,\alpha}(\Omega)} = \sup_{\mathbf{x}\neq\mathbf{y}\in\Omega} |f(\mathbf{x} - f(\mathbf{y})|\ |\mathbf{x} - \mathbf{y}|^{-\alpha}$ the Hölder semi-norm so that $\|f\|_{C^{0,\alpha}(\Omega)} = \|g\|_{C^0(\Omega)} + [g]_{C^{0,\alpha}(\Omega)}$. Then

$$
\begin{aligned}
\|N[e]\|_{C^{0,\alpha}(\Omega)} &= \sup_{x\neq y\in\Omega} \frac{f(\mathbf{x})e(\mathbf{x}) - f(\mathbf{y})e(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{\alpha}} \\
&= \sup_{\mathbf{x}\neq\mathbf{y}\in\Omega} \frac{f(\mathbf{x})(e(\mathbf{x}) - e(\mathbf{y})) + (f(\mathbf{x}) - f(\mathbf{y}))e(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{\alpha}} \\
&\leq \|f\|_{C^0(\Omega)}[e]_{C^{0,\alpha}(\Omega)} + [f]_{C^{0,\alpha}(\Omega)}\|e\|_{C^0(\Omega)} \\
&\leq 2\|f\|_{C^{0,\alpha}(\Omega)}\|e\|_{C^{0,\alpha}(\Omega)}
\end{aligned}
$$

and the linear operator is continuous.

□

A.3.2 Continuity of Composition Operators

Now some composition operator continuity results.

Lemma A.3. *In all cases we consider a map* $\mathbf{f} : \mathbb{R}^N \times \Omega \to \mathbb{R}^M$ *and the Nemytskii operator* $N[\mathbf{u}](\mathbf{x}) = \mathbf{f}(\mathbf{u}(\mathbf{x}), \mathbf{x})$.

1. *Suppose* $\mathbf{f}$ *is Carathéodory and satisfies* $|\mathbf{f}(\mathbf{w},\mathbf{x})| \leq C\,(1+|\mathbf{w}|^p)$ *with* $0 < p \leq q < \infty$ *and* $1 \leq q$ *so that* $r = \frac{q}{p} \in [1,\infty)$ *. Then the map* $N : L^q(\Omega,\mathbb{R}^N) \to L^r(\Omega,\mathbb{R}^M)$ *is continuous.*

2. *Suppose* $\mathbf{f}$ *and* $\nabla_x\mathbf{f}$ *are Carathéodory,* $D_w\mathbf{f}$ *is jointly continuous on* $\mathbb{R}^N \times \overline{\Omega}$*, and* $\int_\Omega |\nabla_x\mathbf{f}(\mathbf{w},\mathbf{x})|^p dV \leq C$ *independent of* $\mathbf{w}$*, then the composition map* $N : W^{1,p}(\Omega,\mathbb{R}^N) \to W^{1,p}(\Omega,\mathbb{R}^M)$ *is continuous.*

3. *Fix* $0 \leq \alpha < 1$ *and suppose* $\mathbf{f}$ *and* $D_w\mathbf{f}$ *are jointly continuous, and* $|\mathbf{f}(\mathbf{w},\mathbf{x})-\mathbf{f}(\mathbf{w},\mathbf{y})| \leq C|\mathbf{x}-\mathbf{y}|^\alpha$ *holding for all* $\mathbf{x},\mathbf{y},\mathbf{w}$*, then the composition map* $N : C^{0,\alpha}(\Omega,\mathbb{R}^N) \to C^{0,\alpha}(\Omega,\mathbb{R}^M)$ *is continuous.*

Proof. 1. Consider a sequence $\mathbf{u}_k \to \mathbf{u}$ in $L^q(\Omega,\mathbb{R}^N)$, then $\mathbf{u}_k(\mathbf{x}) \to \mathbf{u}(\mathbf{x})$ for almost every $\mathbf{x} \in \Omega$ and the sequence is uniformly bounded in $L^q(\Omega,\mathbb{R}^N)$. More precisely there is a compact set $E_\delta \subset \Omega$ such that $\mathbf{u}_k$ is converging uniformly on E_δ with $|\Omega - E_\delta| < \delta$ for any δ by Egorov's theorem. Then $\mathbf{f}(\mathbf{u}_k,\mathbf{x})$ converges uniformly on E_δ, and since uniform convergence on a compact set implies L^r convergence, $\mathbf{f}(\mathbf{u}_k,\mathbf{x})$ converges in $L^r(E_\delta,\mathbb{R}^M)$.

We need one more measure theory fact, which in terms of $\mathbf{u}$ is that for measurable sets $F \subset \Omega$

$$\sup_{|F|\leq\delta} ||\mathbf{u}||_{L^p(F,\mathbb{R}^N)} \to 0 \text{ as } \delta \to 0. \tag{A.5}$$

Suppose to the contrary that there are decreasing sets of measure less then $\frac{1}{n}$ such that $||\mathbf{u}||^p_{L^p(F_n,\mathbb{R}^N)} \geq c$. The functions $|\mathbf{u}|^p\chi_{F_n}$ are dominated by $|\mathbf{u}|^p$ and converging point-wise to zero, so they converge in norm to zero, which contradicts that the norms are bounded below.

We bound the norm separately on E_δ and on the remaining set of small measure.

$$\begin{aligned}\left\|N[\mathbf{u}] - N[\mathbf{u}_k]\right\|_{L^r(\Omega,\mathbb{R}^M)} \leq &\left\|N[\mathbf{u}] - N[\mathbf{u}_k]\right\|_{L^r(E_\delta,\mathbb{R}^M)} \\ &+ \left\|N[\mathbf{u}]\right\|_{L^r(\Omega-E_\delta,\mathbb{R}^M)} + \left\|N[\mathbf{u}_k]\right\|_{L^r(\Omega-E_\delta,\mathbb{R}^M)}\end{aligned} \tag{A.6}$$

and for $\mathbf{w}$ equal to either $\mathbf{u}$ or $\mathbf{u}_k$,

$$\begin{aligned}\left\|N[\mathbf{w}]\right\|^r_{L^r(\Omega-E_\delta,\mathbb{R}^M)} \leq &C \int_{\Omega-E_\delta} (1 + |\mathbf{w}|^p)^r \, dV \\ \leq &2^{r-1} C \left(\delta + \|\mathbf{w}\|^q_{L^q(\Omega,\mathbb{R}^N)}\right) .lnn\end{aligned} \tag{A.7}$$

We also used a simple inequality $(1 + |y|^a)^b \leq 2^{b-1}(1 + |y|^{ab})$ for $b \geq 1$, which is an application of Jensen's inequality (dependent on the convexity of the monomial $|z|^b$). Then continuing from (A.6),

$$\begin{aligned}\left\|N[\mathbf{u}] - N[\mathbf{u}_k]\right\|_{L^r(\Omega,\mathbb{R}^M)} \leq &\left\|N[\mathbf{u}] - N[\mathbf{u}_k]\right\|_{L^r(E_\delta,\mathbb{R}^M)} \\ &+ 2^{r-1} C \|\mathbf{u} - \mathbf{u}_k\|^{q/r}_{L^q(\Omega,\mathbb{R}^N)} + 2^r C \|\mathbf{u}\|_{L^q(\Omega-E_\delta,\mathbb{R}^N)} + 2^{r-1}\delta.\end{aligned}$$

Next we choose the δ for Egorov's theorem to be small enough that $2^r C \|\mathbf{u}\|_{L^q(\Omega-E_\delta,\mathbb{R}^N)} + 2^{r-1}\delta < \epsilon/2$. Then we choose k large enough that $\|N[\mathbf{u}] - N[\mathbf{u}_k]\|_{L^r(\Omega,\mathbb{R}^M)} < \epsilon/4$ and $2^{r-1} C \|\mathbf{u} - \mathbf{u}_k\|^{q/r}_{L^q(\Omega,\mathbb{R}^N)} < \epsilon/4$.

2. This is shown in the same manner of the preceding argument. Similar to the part 2 of Lemma A.2, only the highest derivatives will matter. Given $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^N)$, the embedding $\mathbf{u} \in C(\overline{\Omega}, \mathbb{R}^N)$ implies that the range of functions in a neighborhood of $\mathbf{u}$ is contained in a a compact set $B \subset \mathbb{R}^N$. Restricted

to B, $\mathbf{f}$ is uniformly continuously differentiable. For the terms off of E_δ, we have the bound

$$\int_{\Omega-E_\delta} \left|D_w\mathbf{f}(\mathbf{u}_k,\mathbf{x})\nabla\mathbf{u}_k\right|^p dV \leq \|\mathbf{u}_k\|^p_{W^{1,p}(\Omega-E_\delta,\mathbb{R}^N)}\|D_w\mathbf{f}\|^p_{L^\infty(B\times\Omega,L(\mathbb{R}^N,\mathbb{R}^M))}$$

and the rest of the argument is identical to part 1.

3. It is convenient to first prove continuity in the case that f is polynomial and then approximate a general f by polynomials. This idea is used in [26] and a different approach is taken in [10]. We work with scalar valued functions and the vector valued case follows easily.

   Consider a sequence $\{g_k\}_{k=1}^\infty \subset C^{0,\alpha}(\Omega)$ and assume that $\|g_k\|_{C^{0,\alpha}(\Omega)} \leq C\|g\|_{C^{0,\alpha}(\Omega)}$ and $g_k \to g$. Then for composition with a monomial (with coefficient dependent on $\mathbf{x}$)

$$\begin{aligned}\left\|a_m g^m - a_m g_k^m\right\|_{C^{0,\alpha}(\Omega)} =& \left\|a_m(g-g_k)\sum_{i+j=m-1} g^i g_k^j\right\|_{C^{0,\alpha}(\Omega)}\\ \leq& C\|a_m\|_{C^{0,\alpha}(\Omega)}\|g-g_k\|_{C^{0,\alpha}(\Omega)}\|g\|^{m-1}_{C^{0,\alpha}(\Omega)}.\end{aligned}$$

   It follows that for any polynomial the Nemytskii operator is continuous. Given $g \in C^{0,\alpha}(\Omega)$ we may choose a neighborhood U so that the range of all functions in the neighborhood is in a compact set, B. Let $\epsilon > 0$ and approximate f by a polynomial p so that $\|f-p\|_{C(\overline{\Omega},C^1(B))} + \|f-p\|_{C^{0,\alpha}(\Omega,C^0(B))} \leq \frac{\epsilon}{6\|g\|_{C^{0,\alpha}(\Omega)}}$ by the Weierstrass approximation theorem. Then if $g_k \to g$ for $g_k \in U$,

$$\begin{aligned}\left\|N[g]-N[g_k]\right\|_{C^{0,\alpha}(\Omega)} \leq& \left\|f(g)-p(g)+p(g)-p(g_k)+p(g_k)-f(g_k)\right\|_{C^{0,\alpha}(\Omega)}\\ \leq& \left\|f(g)-p(g)\right\|_{C^{0,\alpha}(\Omega)} + \left\|p(g)-p(g_k)\right\|_{C^{0,\alpha}(\Omega)}\\ &+ \left\|p(g_k)-f(g_k)\right\|_{C^{0,\alpha}(\Omega)}\end{aligned}$$

For the first and last term, let h be g or g_k,

$$\begin{aligned}\left\|f(h,\cdot)-p(h,\cdot)\right\|_{C^{0,\alpha}(\Omega)} &= \sup_{\mathbf{x}\neq\mathbf{y}} \frac{f(h(\mathbf{x}),\mathbf{x})-p(h(\mathbf{x}),\mathbf{x})-f(h(\mathbf{y}),\mathbf{y})+p(h(\mathbf{y}),\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\alpha}} \\ &= \sup_{\mathbf{x}\neq\mathbf{y}} \frac{(f-p)(h(\mathbf{x}),\mathbf{x})-(f-p)(h(\mathbf{y}),\mathbf{x})}{\left|h(\mathbf{x})-h(\mathbf{y})\right|}\frac{\left|h(\mathbf{x})-h(\mathbf{y})\right|}{|\mathbf{x}-\mathbf{y}|^{\alpha}} \\ &\quad + \frac{(f-p)(h(\mathbf{y}),\mathbf{x})-(f-p)(h(\mathbf{y}),\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{\alpha}} \\ &\leq \|f-p\|_{C(\overline{\Omega},C^1(B))}\|h\|_{C^{0,\alpha}(\Omega)}+\|f-p\|_{C^{0,\alpha}(\Omega,C^0(B))}\|h\|_{C^0(\Omega)} \\ &\leq \frac{\epsilon}{3}.\end{aligned}$$

We used the mean value theorem for derivatives to equate the Lipschitz semi-norm of f with the C^1 norm.

By continuity of polynomial composition, we choose k large enough so that the middle term is smaller then $\epsilon/3$, showing convergence and continuity of the composition.

□

A.3.3 Continuous Differentiability

Here we present the main theorem for continuous Fréchet differentiability. More information for Hölder continuity can be found in [26], [10]. For more on differentiability on the L^p spaces there are [7] and [15].

Theorem A.1. *As above, let $\Omega \subset \mathbb{R}^n$ be open and bounded and $N[\mathbf{u}](\mathbf{x}) = \mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})$.*

1. *Suppose* $\mathbf{f}$ *and* $D_w\mathbf{f}$ *are Carathéodory. Suppose also* $|\mathbf{f}(\mathbf{w},\mathbf{x})| \leq C(1+|\mathbf{w}|^p)$, *and* $|D_w\mathbf{f}(\mathbf{w},\mathbf{x})| \leq C(1+|\mathbf{w}|^{p-1})$ *for* $1 < p \leq q < \infty$. *Then with* $r = q/p$, $N : L^q(\Omega,\mathbb{R}^N) \to L^r(\Omega,\mathbb{R}^M)$ *is continuously Fréchet differentiable. This result is generally attributed to Krasnolselskii.*

2. *Suppose* $\mathbf{f}$, $\nabla_x\mathbf{f}$, $D_w\nabla_x\mathbf{f}$ *are all Carathéodory,* $D_w\mathbf{f}$ *and* $D_w^2\mathbf{f}$ *are jointly continuous, and* $\int_\Omega |D_w\nabla_x\mathbf{f}(\mathbf{w},\mathbf{x})|^p dV \leq C$ *for all* $\mathbf{w}$ *with* $p > n$, *then* $N : W^{1,p}(\Omega,\mathbb{R}^N) \to W^{1,p}(\Omega,\mathbb{R}^M)$ *is continuously Fréchet differentiable.*

3. *Suppose* $\mathbf{f}$, $D_w\mathbf{f}$ *and* $D_w^2\mathbf{f}$ *are jointly continuous and* $|\mathbf{f}(\mathbf{w},\mathbf{x}) - \mathbf{f}(\mathbf{w},\mathbf{y})| + |D_w\mathbf{f}(\mathbf{w},\mathbf{x}) - D_w\mathbf{f}(\mathbf{w},\mathbf{y})| \leq C|\mathbf{x}-\mathbf{y}|^\alpha$, *then* $N : C^{0,\alpha}(\Omega,\mathbb{R}^N) \to C^{0,\alpha}(\Omega,\mathbb{R}^M)$ *is continuously Fréchet differentiable.*

In all cases the derivative is given by

$$(DN[\mathbf{u}]\mathbf{h})(\mathbf{x}) = D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\mathbf{h}(x). \tag{A.8}$$

Proof. 1. First we bound the remainder term of (A.1) near $\mathbf{u} \in L^q(\Omega,\mathbb{R}^N)$ using the fundamental theorem of calculus,

$$\begin{aligned}
&\left\|N[\mathbf{u}+\mathbf{h}] - N[\mathbf{u}] - DN[\mathbf{u}]\mathbf{h}\right\|^r_{L^r(\Omega,\mathbb{R}^M)} \\
=& \int_\Omega \left|\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - \mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\cdot\mathbf{h}(\mathbf{x})\right|^r dV \qquad \text{(A.9)} \\
=& \int_\Omega \left|\int_0^1 \left[D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right]\cdot\mathbf{h}(\mathbf{x})\,dt\right|^r dV \\
\leq& \int_0^1 \int_\Omega \left|\left[D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x})) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right]\cdot\mathbf{h}(\mathbf{x})\right|^r dVdt \\
\leq& \|\mathbf{h}\|^r_{L^q(\Omega,\mathbb{R}^N)} \int_0^1 \left(\int_\Omega \left|D_w\mathbf{f}(\mathbf{u}+t\mathbf{h},\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^m dV\right)^{r/m} dt
\end{aligned}$$

where $1/m + 1/q = 1/r$. The dt integral was interchanged with $|\cdot|^r$ by Jensen's inequality as $|\cdot|^r$ is convex for $r \geq 1$, and $[0, 1]$ has measure 1. Then Fubini's theorem allows interchanging the dV and dt integrals as $\mathbf{u}(\mathbf{x}) + t\mathbf{h}(\mathbf{x})$ is jointly measurable in $\mathbf{x}$ and t and so is $D_w\mathbf{f}(\mathbf{u}(\mathbf{x}) + t\mathbf{h}(\mathbf{x}), \mathbf{x}) \cdot \mathbf{h}(\mathbf{x})$. The remainder is bounded by the term

$$\epsilon(\mathbf{h}) = \int_0^1 \left(\int_\Omega \left| D_w\mathbf{f}(\mathbf{u} + t\mathbf{h}, \mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}), \mathbf{x}) \right|^m dV \right)^{r/m} dt$$

and it is necessary to show that $\epsilon(\mathbf{h}) \to 0$ as $\mathbf{h} \to 0$. Now $D_w\mathbf{f}$ satisfies the hypothesis of Lemma A.3 with $q' = q$, $p' = p - 1$ and $r' = m = \frac{q}{p-1}$, so the Nemytskii operator given by $N'[\mathbf{w}](\mathbf{x}) = D_w\mathbf{f}(\mathbf{w}(\mathbf{x}), \mathbf{x})$ is continuous from $L^q(\Omega, \mathbb{R}^N) \to L^m(\Omega, L(\mathbb{R}^N, \mathbb{R}^M))$. Since $|t\mathbf{h}(\mathbf{x})| \leq |\mathbf{h}(\mathbf{x})|$, $N'[\mathbf{u} + t\mathbf{h}] \to N'[\mathbf{u}]$ in $L^m(\Omega, L(\mathbb{R}^N, \mathbb{R}^M))$ and this is uniformly in t so $\epsilon(\mathbf{h})$ goes to zero. This has shown that $N'[\mathbf{u}]$ is the Fréchet derivative, i.e. $(DN[\mathbf{u}]\mathbf{h})(\mathbf{x}) = N'[\mathbf{u}](\mathbf{x})\mathbf{h}(\mathbf{x})$.

For continuity, Lemma A.3 shows that $N' : L^q(\Omega, \mathbb{R}^N) \to L^m(\Omega, L(\mathbb{R}^N, \mathbb{R}^M))$ is continuous, and the the result of Lemma A.2 enables us to equate the norm of $BL(L^q(\Omega, \mathbb{R}^N), L^r(\Omega, \mathbb{R}^M))$ with the $L^m(\Omega, L(\mathbb{R}^N, \mathbb{R}^M))$ norm of $N'[\mathbf{u}]$ and thus the derivative map from $L^q(\Omega, \mathbb{R}^N) \to BL(L^q(\Omega, \mathbb{R}^N), L^r(\Omega, \mathbb{R}^N))$ is continuous.

2. This proof uses the same idea but also makes use of the Sobolev embedding into a Hölder space. Consider the highest derivatives of the remainder and

use the same calculus trick as in equation (A.9),

$$
\begin{aligned}
&\int_\Omega \left|\nabla\Big(N[\mathbf{u}+\mathbf{h}](\mathbf{x}) - N[\mathbf{u}](\mathbf{x}) - (DN[\mathbf{u}]\mathbf{h})(\mathbf{x})\Big)\right|^p dV \\
=&\int_\Omega \Big|[D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})]\nabla\mathbf{h}(\mathbf{x}) \\
&+ \nabla_x\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - \nabla_x\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x}) - \Big(\nabla_x D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\Big)\mathbf{h}(\mathbf{x}) \\
&[D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})]\nabla\mathbf{u}(\mathbf{x}) - (\nabla\mathbf{u}(\mathbf{x}))^\top D_w^2\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\mathbf{h}(\mathbf{x})\Big|^p dV \\
\leq& C\int_\Omega \Big(\left|\nabla\mathbf{h}(\mathbf{x})\right|^p \left|D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p \\
&+ \int_0^1 \left|\mathbf{h}(\mathbf{x})\right|^p \left|\nabla_x D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x}),\mathbf{x}) - \nabla_x D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p dt \\
&+ \int_0^1 \left|\mathbf{h}(\mathbf{x})\right|^p \left|\nabla\mathbf{u}(\mathbf{x})\right|^p \int_0^1 \left|D_w^2\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w^2\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p dt\Big) dV \\
\leq& \|\mathbf{h}\|^p_{W^{1,p}(\Omega,\mathbb{R}^N)}\Big(\int_\Omega \left|D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p dV \\
&+ \int_0^1\int_\Omega \left|\nabla_x D_w\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x}),\mathbf{x}) - \nabla_x D_w\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p dVdt \\
&+ \int_0^t\int_\Omega \left|\nabla\mathbf{u}(\mathbf{x})\right|^p \left|D_w^2\mathbf{f}(\mathbf{u}(\mathbf{x})+t\mathbf{h}(\mathbf{x}),\mathbf{x}) - D_w^2\mathbf{f}(\mathbf{u}(\mathbf{x}),\mathbf{x})\right|^p dVdt\Big).
\end{aligned}
\tag{A.10}
$$

The first two terms of (A.10) is go to zero due to continuity of $D_w\mathbf{f} : W^{1,p}(\Omega,\mathbb{R}^N) \to W^{1,p}(\Omega, L(\mathbb{R}^N,\mathbb{R}^M))$ from part 2 of Lemma A.3. The las term goes to zero because $\nabla\mathbf{u}$ has bounded L^p norm and the composition $D_w^2\mathbf{f} : C^0(\Omega,\mathbb{R}^N) \to C^0(\Omega, BL(\mathbb{R}^N,\mathbb{R}^M))$ is continuous by an application of part of the proof of part 3 of Lemma A.3.

The previous results on continuity immediately show the derivative is continuous.

3. For Hölder spaces the story is not much different. In this case we must

interchange an integral in t of the form

$$
\begin{aligned}
&\left\| \int_0^t \Big(D_w \mathbf{f}(\mathbf{u}(\mathbf{x}) + t\mathbf{h}(\mathbf{x}), \mathbf{x}) - D_w \mathbf{f}(\mathbf{u}(\mathbf{x}), \mathbf{x}) \Big) \mathbf{h}(\mathbf{x}) dt \right\|_{C^{0,\alpha}(\Omega)} \\
\leq &\int_0^t \left\| \Big(D_w \mathbf{f}(\mathbf{u}(\mathbf{x}) + t\mathbf{h}(\mathbf{x}), \mathbf{x}) - D_w \mathbf{f}(\mathbf{u}(\mathbf{x}), \mathbf{x}) \Big) \mathbf{h}(\mathbf{x}) \right\|_{C^{0,\alpha}(\Omega)} dt
\end{aligned} \tag{A.11}
$$

but this is straight forward.

The remainder is bounded by

$$
\epsilon(\mathbf{h}) = \int_0^1 \left\| D_w \mathbf{f}(\mathbf{u}(\mathbf{x}) + t\mathbf{h}(\mathbf{x}), \mathbf{x}) - D_w \mathbf{f}(\mathbf{u}(\mathbf{x}), \mathbf{x}) \right\|_{C^{0,\alpha}(\Omega)} dt
$$

and as $\mathbf{h} \to 0$ the composition result for $D_w \mathbf{f}$ implies $\epsilon(\mathbf{h}) \to 0$ as $\mathbf{h} \to 0$.

For continuity again just use the two continuity results.

□

APPENDIX B

DETAILS OF THE LAGRANGE MULTIPLIER THEOREM AND IMPLICIT FUNCTION THEOREM

All spaces are assumed to be Banach spaces. See chapter A for the definition of Gâteaux differentiable, Definition A.3, and (continuously) Fréchet differentiable, Definition A.1 (A.2).

The motivation of this chapter is to develop the tools to characterize the first order optimality criterion for constrained minimum of suitably differentiable functionals. Here is the most basic first order optimality condition.

Theorem B.1. *Suppose $E : X \to \mathbb{R}$ has a minimum at x^* and has a Gâteaux derivative, $L \in X^*$, at x^*. Then $\langle L, h\rangle = 0$ for all $h \in X$.*

Proof. Fix $h \in X$, then for $\epsilon > 0$

$$\frac{E[x^*] - E[x^* + \epsilon h]}{\epsilon} \leq 0$$

as x^* is a minimum, and also if $\epsilon < 0$ then

$$\frac{E[x^*] - E[x^* + \epsilon h]}{\epsilon} \geq 0.$$

Since E is Gâteaux differentiable the limit, $\langle L, h\rangle$, exists as ϵ approaches and therefore must be 0. □

Theorem B.1 requires that the path $x^* + \epsilon h$ be admissible for small ϵ. With constraints, we must find admissible variations that satisfy the constraints and agree with $x^* + \epsilon h$ to first order in ϵ when h satisfies the linearized constraint at x^*.

Question 1: Is it sufficient to only assume the objective function is Gâteaux differentiable for the Lagrange Multiplier theorem (if the constraint is still C^1 and the linearization is surjective).

Here is a potential (but incorrect) statement of Lagrange Multiplier Theorem (LMT) for the constrained minimization problem to minimize $E : X \to \mathbb{R}$ subject to $G[x] = 0$ for $G : X \to Z$.

Theorem B.2. *(FALSE) Suppose that E has a constrained minimum at x_0 and G is C^1 in a neighborhood of x^* and $DG[x^*]$ is surjective. Suppose that E is Gâteaux differentiable at x^*, with derivative L. Then there is $p \in Z^*$ such that for all $h \in X$,*

$$\langle L, h\rangle - \langle p, DG[x^*]h\rangle = 0.$$

There is a counter example with $X = \mathbb{R}^2$. Let $Z = \mathbb{R}$, $G[\mathbf{x}] = \mathbf{x} \cdot \mathbf{e}_2 - (\mathbf{x} \cdot \mathbf{e}_1)^2$, and

$$E[\mathbf{x}] = \left\{ \begin{array}{r|l} 2|\mathbf{x}| + \mathbf{x} \cdot \mathbf{e}_1 & G[\mathbf{x}] = 0 \\ \mathbf{x} \cdot \mathbf{e}_1 & G[\mathbf{x}] \neq 0. \end{array} \right.$$

The constrained problem has a global minimum at $\mathbf{x}^* = \mathbf{0}$. The Gâteaux derivative of E exists at this point and is $DE[\mathbf{0}] = \mathbf{e}_1$. The derivative of G is $DG[\mathbf{0}] = \mathbf{e}_2$, which is linearly independent from $\mathbf{e}_1$, so there does not exist a Lagrange multiplier.

It could also be assumed that E is continuous and everywhere Gâteaux differentiable, but a further example has the same phenomenon with $E \in C^1(\mathbb{R}^2 \backslash \{\mathbf{0}\}) \cap$

$C(\mathbb{R}^2)$ and Gâteux differentiable at the origin,

$$E[\mathbf{x}] = \begin{cases} \frac{4|\mathbf{x}|^3(\mathbf{x}\cdot\mathbf{e}_2)}{(\mathbf{x}\cdot\mathbf{e}_1)^4+(\mathbf{x}\cdot\mathbf{e}_2)^2} + \mathbf{x}\cdot\mathbf{e}_1 & \mathbf{x} \neq \mathbf{0} \\ 0 & \mathbf{x} = \mathbf{0}. \end{cases}$$

Again we can check that $\mathbf{0}$ is a global constrained minimum and $DE[\mathbf{0}] = \mathbf{e}_1$ so there is no Lagrange multiplier.

Question 2: Why assume that the linearized constraint is surjective and how is the surjective Implicit Function Theorem (IFT) used for the proof of LMT.

The surjectivity assumption is also necessary in finite dimensions. Let $X = \mathbb{R}^2$ and consider the constraints $G_1[\mathbf{x}] = \mathbf{x}\cdot\mathbf{e}_2 - (\mathbf{x}\cdot\mathbf{e}_1)^2$ and $G_2[\mathbf{x}] = \mathbf{x}\cdot\mathbf{e}_2 - 2(\mathbf{x}\cdot\mathbf{e}_1)^2$. The constraints $G_1[\mathbf{x}] = G_2[\mathbf{x}] = 0$ imply that $\mathbf{x} = \mathbf{0}$, so the single feasible point is always a minimum. However at $\mathbf{0}$ the gradients of G_1 and G_2 are both $\mathbf{e}_2$. Since we have made no assumptions on the objective function it is not necessary that the the derivatives of E vanish in the $\mathbf{e}_1$ direction and there may be no Lagrange multiplier.

The IFT is often presented as:

Theorem B.3. *Let $g : W \times Y \to Z$ be C^1 in a neighborhood of (w_0, y_0), $g[w_0, y_0] = 0$, and suppose that $D_Y g[w_0, y_0] : Y \to Z$ is invertible. Then there is a neighborhood, $N_W \times N_Y$, of (w_0, y_0) and a C^1 map $\phi : N_W \to N_Y$ such that $g[w, \phi[w]] = 0$ and if $(w, y) \in N_W \times N_Y$ satisfies $g[w, y] = 0$, then $y = \phi[w]$.*

It is also easy to see that

$$D\phi[w_0] = -D_Y g[w_0, y_0]^{-1} D_W g[w_0, y_0] \tag{B.1}$$

because

$$0 = \frac{d}{d\epsilon} g[w_0 + \epsilon h, \phi[w_0 + \epsilon h]] = D_W g[w_0, y_0]h + D_Y g[w_0, y_0] D\phi[w_0]h.$$

If instead $g : X \to Z$ and we only assume that $Dg[x_0]$ is surjective, then it is natural to think that $X = W \times Y$ with $x_0 = (w_0, y_0)$, $W = \ker(Dg[x_0])$, and a subspace $Y \subset X$ on which $D_Y g[w_0, y_0] : Y \to Z$ is invertible. If X is a Hilbert space (for example if X is finite dimensional) then choose Y to be the orthogonal complement of the null space of $Dg[x_0]$ and the IFT holds. For a general Banach space, i.e. $L^p(\Omega)$ for $p \neq 2$, not every closed subspace has a topological complement, which is needed for this decomposition work.

The surjective IFT has a slightly weaker hypothesis and a slightly weaker result.

Theorem B.4. *Suppose $g : W \times Y \to Z$ is C^1 in a neighborhood of (w_0, y_0), $g[w_0, y_0] = 0$ and $D_Y g[w_0, y_0] : Y \to Z$ is surjective. Then there is a neighborhood $N_W \times N_Y$ of (w_0, y_0) and a continuous map $\phi : N_W \to N_Y$ such that for all $w \in N_W$, $g[w, \phi[w]] = 0$, and there is a constant C such that*

$$\left\|\phi[w] - y_0\right\|_Y \leq C \left\|D_Y g[w_0, y_0](\phi[w] - y_0)\right\|_Z. \tag{B.2}$$

Furthermore, if $h \in \ker(D_W g[w_0, y_0])$, then $\phi[w_0 + \epsilon h]$ is differentiable with respect to ϵ at $\epsilon = 0$ and $\frac{d}{d\epsilon}\phi[w_0 + \epsilon h]|_{\epsilon=0} = 0$.

The difference with Theorem B.3 is that we have lost uniqueness of solutions as well as continuous differentiability of the solution map.

B.0.1 Lagrange Mulitiplier Theorem

First we prove the LMT using Theorem B.4 and point out where B.3 could be used with stronger assumptions. Then we will give a proof of B.4. These proofs may be found in [35] or other sources.

Theorem B.5. *Suppose that $G : X \to Z$ and $E : X \to \mathbb{R}$ are C^1 in a neighborhood U of x^*. Suppose also that $G[x^*] = 0$, $DG[x^*]$ is surjective, and x^* is a local minimum, i.e. $E[x] \geq E[x^*]$ for all $x \in U$ such that $G[x] = 0$. Then there exists $p \in Z^*$ such that*

$$\langle DE[x^*], h\rangle - \langle p, DG[x^*]h\rangle = 0 \ \forall \ h \in X. \tag{B.3}$$

Proof. We consider a direction $h \in \ker(DG[x^*])$ and show that $\langle DE[x^*], h\rangle = 0$. Once this property is verified, then $DE[x^*]$ viewed as a linear functional of X satisfies

$$DE[x^*] \in \ker\left(DG[x^*]\right)^{\perp}. \tag{B.4}$$

The closed range theorem implies that, since $DG[x^*]$ has closed range (it is surjective), $\ker(DG[x^*])^{\perp} = R(DG[x^*]^{\top})$ where $DG[x^*]^{\top} : Z^* \to X^*$ is the adjoint. Thus there is a linear functional, $p \in Z^*$, such that $DE[x^*] = DG[x^*]^{\top} p$. Reinterpreting with the dual pairing, we have shown Equation (B.3).

Let us first consider the proof that $\langle DE[x^*], h\rangle = 0$ with the stronger assumption that X decomposes as $W \times Y$ where $W = \ker(DG[x^*])$ and Y is a closed subspace on which $D_Y G[x^*]$ is invertible. Let $x^* = (w_0, y_0)$ and any $h \in \ker(DG[x^*])$ decomposes as $(h_W, 0)$. We apply Theorem B.3 and consider a family of solutions to the constraint equations corresponding to $x(\epsilon) = (w_0 + \epsilon h_W, \phi(w_0 + \epsilon h_W))$. Equation B.1 implies $D\phi[w_0]h_W = 0$. By optimality of x^* we know that

$$
\begin{aligned}
0 =& \frac{d}{d\epsilon} E[x(\epsilon)] \\
=& DE[x^*](h_W, D\phi[w_0]h_W) \\
=& DE[x^*]h.
\end{aligned}
$$

We can instead apply the surjective IFT to

$$g[\epsilon, y] = G[x^* + \epsilon h + y]$$

mapping $g : \mathbb{R} \times X \to Z$. The result of Theorem B.4 (identify $W = \mathbb{R}$ and $Y = X$) is a continuous map $\phi(\epsilon) : \mathbb{R} \to X$. Since $h \in \ker(DG[x^*])$, $D_\epsilon g[0, 0] = 0$ so by Theorem B.4, ϕ is differentiable at 0 and $\frac{d}{d\epsilon}\phi(0) = 0$.

The chain rule (using continuous differentiability of E) implies that

$$
\begin{aligned}
0 =& \frac{d}{d\epsilon} E[x^* + \epsilon h + \phi(\epsilon)] \\
=& DE[x^*]h + DE[x^*]\frac{d}{d\epsilon}\phi(0) \\
=& DE[x^*]h.
\end{aligned}
$$

□

B.0.2 Implicit Function Theorem

Here is a proof of Theorem B.3.

Proof. It follows from the inverse mapping theorem that $D_Y g[w,y]^{-1} : Z \to W$ is defined and continuous in a neighborhood of (w_0, y_0) and suppose $\|D_Y g[w_0, y_0]^{-1}\|_{BL(Z,Y)} \le C_1$. Let

$$A[w,y] = y - D_Y g[w_0, y_0]^{-1} g[w,y]$$

and consider the fixed point equation equivalent to $g[w,y] = 0$,

$$y = A[w,y]. \tag{B.5}$$

We calculate using the mean value theorem and some $y' = (1-t)y_0 + ty_1$

$$\begin{aligned} \left\|A[w,y] - y_0\right\|_Y \le& C_1 \left\|D_Y g[w_0,y_0](y-y_0) - g[w,y]\right\|_Z \\ \le& C_1 \left(\left\|D_Y g[w_0,y_0] - D_Y g[w,y']\right\|_{BL(Y,Z)} \left\|y - y_0\right\| + \left\|g[w,y_0]\right\|_Z \right), \end{aligned} \tag{B.6}$$

and for $y' = (1-t)y_1 + ty_2$ with $t \in [0,1]$

$$\begin{aligned} \left\|A[w,y_1] - A[w,y_2]\right\|_Y \le& C_1 \left\| - D_Y g[w_0,y_0] - D_Y g[w_0,y_0](y_2 - y_1) - g[w,y_1] + g[w,y_2]\right\|_Z \\ \le& C_1 \left\|D_Y g[w_0,y_0] - D_Y g[w,y']\right\|_{BL(Y,Z)} \|y_1 - y_2\|_Y. \end{aligned} \tag{B.7}$$

For $r, \rho > 0$ define

$$\begin{aligned} N_W =& \{w \in W : \ \|w - w_0\|_W \le \rho\}, \\ N_Y =& \{y \in Y : \ \|y - y_0\|_Y \le r\}. \end{aligned}$$

Choose ρ and r small enough that $\|A[w,y] - y_0\|_Y \le \frac{r}{2}$ (choosing ρ first and noting that the derivatives are converging by continuous differentiability of g), and $\|A[w,y_1] - A[w,y_2]\|_Y \le \frac{1}{2}\|y_1 - y_2\|$. By the Banach contraction mapping theorem, there is a unique solution to (B.5) in $N_W \times N_Y$ and call it $(w, \phi[w])$. To show that ϕ is continuous we check

$$\begin{aligned}
\left\|\phi[w_1] - \phi[w_2]\right\|_Y \leq & \left\|A[w_1, \phi[w_1]] - A[w_2, \phi[w_2]]\right\|_Y \\
\leq & \left\|A[w_1, \phi[w_1]] - A[w_1, \phi[w_2]]\right\|_Y + \left\|A[w_1, \phi[w_2]] - A[w_2, \phi[w_2]]\right\|_Y \\
\leq & \frac{1}{2}\left\|\phi[w_1] - \phi[w_2]\right\|_Y + \left\|A[w_1, \phi[w_2]] - A[w_2, \phi[w_2]]\right\|_Y .
\end{aligned}$$

Next we show that

$$D\phi[w] = -D_Y g[w, \phi[w]]^{-1} D_W g[w, \phi[w]]$$

is the Fréchet derivative and is continuous. First we calculate

$$\begin{aligned}
0 = & \left\|g[w+h, \phi[w+h]] - g[w, \phi[w+h]]\right\|_Z \\
= & \left\|D_W g[w', \phi[w']]h + D_Y g[w', \phi[w']](\phi[w+h] - \phi[w])\right\|_Z
\end{aligned}$$

for some $w' \to w$ as $h \to 0$. Then

$$\begin{aligned}
& \frac{\left\|\phi[w+h] - \phi[w] + D_Y g[w, \phi[w]]^{-1} D_W g[w, \phi[w]]h\right\|_Y}{\|h\|_W} \\
\leq & \left\| - D_Y g[w', \phi[w']]^{-1} D_W g[w', \phi[w']] + D_Y g[w, \phi[w]]^{-1} D_W g[w, \phi[w]]\right\|_{BL(W,Y)}
\end{aligned}$$

By continuity of the derivatives and continuity of ϕ we see that this approaches zero as $h \to 0$. It is easy to verify that the derivative is continuous. □

Now let us give a proof of Theorem B.4. The proof is similar to the argument above, except we cannot use the inverse map.

Proof. There is a continuous linear map $B : Z \to Y$ such that $\|B\| \le C_1$ and $D_Y g[w_0, y_0]Bz = z$. Although we do not quite get a contraction mapping we are able to obtain the existence of a fixed point by the same technique and get the inequality (B.2). Let $f[w, y] = D_Y g[w_0, y_0](y - y_0) - g[w, y]$ and then $g[w, y] = 0$ is equivalent to $D_Y g[w_0, y_0](y - y_0) = f[w, y]$. In the same manner as equations (B.6) and (B.7) we find

$$\begin{aligned}\left\|f[w,y]\right\|_Z \le& \left\|D_Y g[w_0,y_0] - D_Y g[w,y']\right\|_{BL(Y,Z)} \|y - y_0\| + \left\|g[w,y_0]\right\|_Z \\ \le& \frac{r}{2C_1}\end{aligned}$$

and

$$\begin{aligned}\left\|f[w,y_1] - f[w,y_2]\right\|_Y \le& \left\|D_Y g[w_0,y_0] - D_Y g[w,y']\right\|_{BL(Y,Z)} \|y_1 - y_2\|_Y, \\ \le& \frac{1}{2C_1}\|y_1 - y_2\|_Y\end{aligned}$$

in a neighborhood $N_W \times N_Y$ for balls of radius ρ and r respectively centered at (w_0, y_0). Fixing $w \in N_W$ let

$$y_{i+1} = y_0 + Bf[w, y_i] \; i \in \{0, 1, \ldots\}.$$

We find that $\|y_i - y_0\|_Y \le \frac{r}{2}$ for all i and $\|y_{i+1} - y_i\| \le \frac{1}{2}\|y_i - y_{i-1}\|$ thus it is a Cauchy sequence and converges to some $y \in N_Y$. The limit satisfies $g[w, y] = 0$ so we call $y = \phi[w]$ and

$$\begin{aligned}\left\|\phi[w] - y_0\right\|_Y \le& \limsup_{i\to\infty} \left\|Bf[w,y_i]\right\|_Y \\ \le& C_1\left\|D_Y g[w_0,y_0](\phi[w] - y_0)\right\|_Z.\end{aligned}$$

To show continuity, consider $w_1, w_2 \in N_W$ and

$$\begin{aligned}
\left\|\phi[w_1] - \phi[w]_2\right\|_Y =& \left\|Bf[w_1, \phi[w_1]] - Bf[w_2, \phi[w_2]]\right\|_Y \\
\leq& C_1\Big(\left\|f[w_1, \phi[w_1]] - f[w_1, \phi[w_2]]\right\|_Z + \left\|f[w_1, \phi[w_2]] - f[w_2, \phi[w_2]]\right\|_Z\Big) \\
\leq& \frac{1}{2}\left\|\phi[w_1] - \phi[w]_2\right\|_Z + C_1\left\|f[w_1, \phi[w_2]] - f[w_2, \phi[w_2]]\right\|_Z. \qquad \text{(B.8)}
\end{aligned}$$

Since $f[w, y]$ is continuous with respect to w for each $y \in N_Y$, we conclude that $\phi[w]$ is continuous.

To show differentiability for $h \in \ker(D_W g[w_0, y_0])$,

$$\begin{aligned}
0 =& \lim_{\epsilon \to 0} \frac{\left\|g[w_0 + \epsilon h, \phi[w_0 + \epsilon h]]\right\|_Z}{\epsilon} \\
=& \lim_{\epsilon \to 0} \left\|D_W g[w_0, y_0]h + \frac{1}{\epsilon} D_Y g[w_0, y_0](\phi[w_0 + \epsilon h] - y_0)\right\|_Z \\
=& \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left\|Dg[w_0, y_0](\phi[w_0 + \epsilon h] - y_0)\right\|_Z
\end{aligned}$$

and by inequality (B.2),

$$\lim_{\epsilon \to 0} \frac{\left\|\phi[w_0 + \epsilon h] - y_0\right\|_Y}{\epsilon} = 0 \qquad \text{(B.9)}$$

so $\frac{d}{d\epsilon}\phi[w_0 + \epsilon h]|_{\epsilon=0} = 0$.

□

APPENDIX C
LIPSCHITZ DOMAIN BOUNDARY ANALYSIS

We assume $\Omega \subset \mathbb{E}^3$ is open, bounded and connected, and $\overline{\Omega}$ is the closure of Ω. When dealing with the boundary of $\overline{\Omega}$, we defined the derivable interior cone $K_x\overline{\Omega}$ in (3.26) for $\mathbf{x} \in \overline{\Omega}$ as

$$K_x\overline{\Omega} = \left\{\mathbf{w}_0 \in \mathbb{E}^3 : \exists\, \epsilon_1 > 0 \text{ and } \mathbf{w} \in C([0,\epsilon_1),\overline{\Omega}) \text{ s.t. } \mathbf{w}(0) = \mathbf{x} \text{ and } \mathbf{w}'(0) = \mathbf{w}_0\right\}.$$

The convex sub-cone $\hat{K}_x\overline{\Omega}$ was defined in (3.28) for $\mathbf{x} \in \overline{\Omega}$ by

$$\hat{K}_x\overline{\Omega} = \left\{\mathbf{w} \in \mathbb{E}^3 : \exists\, \mathbf{z} \in C(\overline{\Omega},\mathbb{E}^3) \text{ s.t. } \mathbf{z}(\mathbf{x}) = \mathbf{w}, \text{ and } \mathbf{z}(\mathbf{y}) \in K_y\overline{\Omega}\ \forall\ \mathbf{y} \in \overline{\Omega}\right\}.$$

See [34] for an alternate definition of $\hat{K}_x\overline{\Omega}$, and the elegant duality with normal cones.

In the case that $\partial\Omega$ is C^1, then $\hat{K}_x\overline{\Omega} = K_x\overline{\Omega} = \{\mathbf{w} \in \mathbb{E}^3 : \mathbf{w}\cdot\boldsymbol{\nu} \le 0\}$ for $\boldsymbol{\nu}$ the outward unit normal vector of $\overline{\Omega}$ at $\mathbf{x}$. This is easy to see from how the cones transform under the C^1 coordinate charts.

Lemma C.1. *The cone $\hat{K}_x\overline{\Omega}$ is convex. Furthermore, if $\partial\Omega$ is (weakly) Lipschitz and $\mathbf{j} \in C(\partial\Omega,\mathbb{E}^3)$ with $\mathbf{j}(\mathbf{x}) \in K_x\overline{\Omega}$ for all $\mathbf{x} \in \partial\Omega$, then $\mathbf{j}(\mathbf{x}) \in \hat{K}_x\overline{\Omega}$ for all $\mathbf{x} \in \partial\Omega$.*

Proof. It is clear that $\hat{K}_x\overline{\Omega}$ is a cone, i.e. closed under positive scalar multiplication. To show convexity it suffices to show that if $\{\mathbf{w}_1,\mathbf{w}_2\} \subset \hat{K}_x\overline{\Omega}$ then $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 \in \hat{K}_x\overline{\Omega}$. This follows if we show that the sum of any two continuous interior vector fields, $\mathbf{z}_1$ and $\mathbf{z}_2$, is interior. Let $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2 \in C(\overline{\Omega},\mathbb{E}^3)$ and we must show that $\mathbf{z}(\mathbf{y}) \in K_y\overline{\Omega}$

for all $\mathbf{y} \in \overline{\Omega}$. Fix $\mathbf{y} \in \overline{\Omega}$ and let $\mathbf{v}_1 \in C([0, \epsilon_1), \overline{\Omega})$ satisfy $\mathbf{v}_1(0) = \mathbf{y}$ and $\mathbf{v}_1'(0) = \mathbf{z}_1(\mathbf{y})$. Then let $\mathbf{v}_2^\epsilon \in C([0, \epsilon_2), \overline{\Omega})$ satisfy $\mathbf{v}_2^\epsilon(0) = \mathbf{v}_1(\epsilon)$ and $\mathbf{v}_2^{\epsilon\prime} = \mathbf{z}_2(\mathbf{v}_1(\epsilon))$ such that $\mathbf{v}_2^\epsilon(\epsilon_3)$ is continuous in ϵ for ϵ_3 fixed. Then $\mathbf{v}(\epsilon) = \mathbf{v}_2^\epsilon(\epsilon)$ is continuous in a neighborhood of $\epsilon = 0$, $\mathbf{v}(0) = \mathbf{y}$ and

$$\begin{aligned}\mathbf{v}'(0) =& \frac{d}{d\epsilon}\Big[\mathbf{v}_2^\epsilon(0) + \mathbf{v}_2^0(\epsilon)\Big]_{\epsilon=0} \\ =& \mathbf{z}(\mathbf{y}) \in K_y\overline{\Omega}.\end{aligned}$$

The second statement of the lemma follows from the definition of $\hat{K}_x\overline{\Omega}$ after extending $\mathbf{j}$ to a continuous vector field on $\overline{\Omega}$ using the Lipschitz coordinate charts. □

Definition C.1. *We say that $\overline{\Omega}$ satisfies the* cone condition *at $\mathbf{x} \in \overline{\Omega}$ if there is an open neighborhood U containing $\mathbf{x}$, a cone C with non-empty interior, and $\delta > 0$, such that for $C^\delta = \{\mathbf{w} \in C : |\mathbf{w}| < \delta\}$, $\mathbf{y} + C^\delta \subset \overline{\Omega}$ for all $\mathbf{y} \in \overline{\Omega} \cap U$.*

Lemma C.2. *The cone $\hat{K}_x\overline{\Omega}$ has an interior point for every $\mathbf{x} \in \overline{\Omega}$ if and only if $\overline{\Omega}$ satisfies the cone condition at $\mathbf{x}$. In particular, if C is a closed cone in $\{\mathbf{0}\} \cup \operatorname{int} \hat{K}_x\overline{\Omega}$, then there is $r > 0$ and $\delta > 0$ such that*

$$\cup_{\mathbf{y} \in B(\mathbf{x}, r, \overline{\Omega})}\{\mathbf{y} + C^\delta\} \subset \overline{\Omega}. \tag{C.1}$$

Proof. Suppose the cone condition is satisfied at $\mathbf{x}$. Then C from the cone condition is a subset of $\hat{K}_x\overline{\Omega}$, thus $\hat{K}_x\overline{\Omega}$ has non-empty interior.

Now suppose that $\hat{K}_x\overline{\Omega}$ has non-empty interior at $\mathbf{x}$. There is a closed finite cone, $C \subset \{\mathbf{0}\} \cup \operatorname{int} \hat{K}_x\overline{\Omega}$. We claim there is $r > 0$ such that for $\mathbf{y} \in B(\mathbf{x}, r, \overline{\Omega})$, $C \in$

int $\hat{K}_y\overline{\Omega}$. There is $\mathbf{z} \in C(\overline{\Omega} \times A, \mathbb{E}^3)$ with $\mathbf{z}(\mathbf{x}, \mathbf{w}) = \mathbf{w}$ or each $\mathbf{w} \in A = C \cap B(\mathbf{0}, 1)$. Then $\mathbf{z}$ extends to a neighborhood of A contained in $\hat{K}_x\overline{\Omega}$. Continuity implies that there is a ball of radius r such that for $\mathbf{y} \in B(\mathbf{x}, r, \overline{\Omega})$, $C \cap B(\mathbf{0}, 1) \subset \mathbf{z}(\mathbf{y}, A)$ hence $C \in \hat{K}_y\overline{\Omega}$.

It also follows from the definition of $K_x\overline{\Omega}$ and openness of Ω that if $C \subset \operatorname{int} \hat{K}_y\overline{\Omega}$ then there is $\delta > 0$ such that $\mathbf{y} + C^\delta \subset \overline{\Omega}$. □

Definition C.2. *The boundary of an open subset of $\mathbb{E}^3$, $\partial\Omega$, is strongly Lipschitz at $\mathbf{x}$ (or just Lipschitz) if there is a two-dimensional ball, B_r, of radius $r > 0$ and centered at the origin, $\phi \in \operatorname{Lip}(\overline{B}_r)$, and an orthonormal basis $\{\mathbf{b}_i\}_{i=1}^3$ such that*

$$B(\mathbf{x}, r, \overline{\Omega}) = \Big\{\mathbf{y} \in B(\mathbf{x}, r) : \mathbf{b}_3 \cdot (\mathbf{y} - \mathbf{x}) \leq \phi(\mathbf{b}_1 \cdot (\mathbf{y} - \mathbf{x}), \mathbf{b}_2 \cdot (\mathbf{y} - \mathbf{x}))\Big\}.$$

The following equivalence is shown in [20] and we sketch a proof.

Lemma C.3. *The boundary of Ω is (strongly) Lipschitz at $\mathbf{x}$ if and only if $\overline{\Omega}$ satisfies the cone condition at $\mathbf{x}$.*

Proof. Suppose that $\overline{\Omega}$ satisfies the cone condition at $\mathbf{x}$. Let $r > 0$ and C be a finite closed cone in the interior of $\hat{K}_y\overline{\Omega}$ for $\mathbf{y} \in B(\mathbf{x}, r, \overline{\Omega})$. We select an orthonormal basis such that $-\mathbf{b}_3 \in \operatorname{int} C$. Then define $\phi(a_1, a_2)$ such that $(\mathbf{x} + a_1\mathbf{b}_1, \mathbf{x} + a_2\mathbf{b}_2, \mathbf{x} + \phi(a_1, a_1)) \in \partial\Omega$. Then ϕ is locally uniquely defined and Lipschitz because of the cone condition.

Now suppose that $\overline{\Omega}$ is represented by the graph of a Lipschitz function, ϕ, at $\mathbf{x}$. Let K be the Lipschitz constant of ϕ. Then the cone condition is satisfied for C generated by the ball of radius $K^{-1}/2$ around $-\mathbf{b}_3$. □

We use the following lemma to extend the portion of the boundary with displacement boundary conditions in Lemma 2.6.

Lemma C.4. *Suppose $\partial\Omega$ is (strongly) Lipschitz and $\Gamma \subset \Omega$ is $W^{2,p}$ in the sense of assumptions* **M** *in Section 2.0.2. Then for any $a > 0$, there exists an open, connected domain Υ with $W^{2,p}$ boundary such that $\Gamma \subset \partial\Upsilon$, and $d(\mathbf{x}, \overline{\Omega}) \leq a$ for all $\mathbf{x} \in \Upsilon$.*

Proof. Let $\gamma > 0$ be a constant such that for any point $\mathbf{x} \in \partial\Omega$, there is a finite interior cone, C^δ, with $\mathbf{y} + C^\delta \subset \overline{\Omega}$ whenever $|\mathbf{y} - \mathbf{x}| < \gamma$. Then we cover $\Gamma \cap \overline{\partial\Omega\backslash\Gamma}$ by balls, $\{B(\mathbf{x}^i, \gamma/3)\}_{i=1}^{M_1}$, and cover $\partial\Omega\backslash(\Gamma \cup_{i=1}^{M_1} B(\mathbf{x}^i, \gamma/3))$ by balls, $\{B(\mathbf{x}^j, \gamma/3)\}_{j=M_1+1}^{M_2}$, such that $\Gamma \cap (\cup_{j=M_1+1}^{M_2} B(\mathbf{x}^j, \gamma/3)) = \emptyset$.

We smooth the boundary recursively in i with $\Omega^0 = \Omega$. By Lemma C.3, we represent $\partial\Omega^{i-1}$ by the graph of a function in a neighborhood of $\mathbf{x}^i$. We choose $\{\mathbf{b}_i\}_{i=1}^3$ to be an orthonormal basis, let B_γ denote a two-dimensional ball of radius γ centered at the origin, and $\phi^i \in \mathrm{Lip}(B_\gamma)$ such that

$$\Omega^{i-1} \cap B(\mathbf{x}^i, \gamma) = \Big\{\mathbf{y} \in B(\mathbf{x}^i, \gamma) : (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_3 < \phi^i\big((\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_1, (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_2\big)\Big\}.$$

If $\mathbf{x}^i \in \Gamma$ then we also have $\hat{\phi}^i \in W^{2,p}(B_\gamma)$ such that

$$\Gamma \cap B(\mathbf{x}^i, \gamma) = \Big\{\mathbf{y} \in B(\mathbf{x}^i, \gamma) : (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_3 = \phi^i\big((\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_1, (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_2\big)\Big\}.$$

We approximate ϕ^i by $\tilde{\phi}^i$ such that $\tilde{\phi}^i \in W^{2,p}(B_{\gamma/3})$, $\tilde{\phi}^i = \phi^i$ on $B_\gamma\backslash B_{2\gamma/3}$, and $|\tilde{\phi}^i - \phi|_\infty < a/N_i$ for N_i the number of balls intersecting $B(\mathbf{x}^i, \gamma)$. Furthermore, we require that $\tilde{\phi}^i$ remains a graph on $B(\mathbf{x}^j, \gamma)$ if it intersects $B(\mathbf{x}^i, \gamma)$, i.e. that $\mathrm{Lip}(\tilde{\phi}^i) \leq (\mathbf{b}_\alpha^j \cdot \mathbf{b}_3^i)^{-1}$ for $\alpha \in \{1, 2\}$, as well as that $\tilde{\phi}^i$ agrees with $\hat{\phi}^i$ for points in a neighborhood of Γ. Now define Ω^i to agree with Ω outside of $B(\mathbf{x}^i, \gamma)$ and let

$$\Omega^i \cap B(\mathbf{x}^i, \gamma) = \left\{\mathbf{y} \in B(\mathbf{x}^i, \gamma) : (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_3 < \tilde{\phi}^i((\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_1, (\mathbf{y} - \mathbf{x}^i) \cdot \mathbf{b}_2)\right\}.$$

The boundary of Ω^i is smooth in $\cup_{j=1}^i B(\mathbf{x}^i, \gamma/3)$ and stays within distance a of $\overline{\Omega}$. Since we do not alter Γ, Ω^{M_2} satisfies the requirements of the lemma. □

Lemma C.5. *Suppose $\Omega \subset \mathbb{E}^3$ satisfies the uniform cone condition, and $\Gamma \subset \partial\Omega$ is closed with $W^{2,p}$ coordinate charts on a neighborhood of Γ. Then there exists $\mathbf{j} \in W^{2,p}(\overline{\Omega}, \mathbb{E}^3)$ such that for every $\mathbf{x} \in \overline{\Omega}\backslash\Gamma$, $\mathbf{j}(\mathbf{x}) \in \operatorname{int} K_x\overline{\Omega}$, and $\mathbf{j}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Gamma$.*

Proof. By the cone condition there is a cover of $\partial\Omega$ by open sets O_i, and finite cones C_i^δ such that $\mathbf{x} + C_i^\delta \subset \overline{\Omega}$ for all $\mathbf{x} \in O_i$. By compactness we may restrict to a finite collection of the sets. We suppose that each cone is of the form $C_i^\delta = \{t\mathbf{v} : t \in [0, \delta],\ |\mathbf{v} - \mathbf{j}_i| \leq \delta\}$ for $\delta > 0$. Clearly $\mathbf{j}_i \in \operatorname{int} \hat{K}_x\overline{\Omega}$ for $\mathbf{x} \in O_i$. Let ψ_i be a partition of unity subordinate to O_i. Let $\eta \in W^{2,p}(\mathbb{E}^3)$ be a function such that $\eta(\mathbf{x}) > 0$ for $\mathbf{x} \in \partial\Omega\backslash\Gamma$, $\eta(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma$ and η has compact support in $\cup O_i$. η may be constructed with coordinate charts in the neighborhood of Γ. Now let $\mathbf{j}(\mathbf{x}) = \eta(\mathbf{x}) \sum \psi_i(\mathbf{x})\mathbf{j}_i$, where the sum is taken over all i such that $\mathbf{x} \in O_i$. Then $\mathbf{j} \in W^{2,p}(\overline{\Omega}, \mathbb{E}^3)$ and $\mathbf{j}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in \Gamma$. That $\mathbf{j}(\mathbf{x}) \in \operatorname{int} K_x\overline{\Omega}$ for $\mathbf{x} \in \overline{\Omega}\backslash\Gamma$ follows from convexity of $\hat{K}_x\overline{\Omega}$, Lemma C.1, because $\mathbf{j}$ is a positive scalar times a convex combination of vectors in the interior of $\hat{K}_x\overline{\Omega}$. □

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